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Error analysis of boundary integral methods

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Many of the boundary value problems traditionally cast as partial differential equations can be reformulated as integral equations over the boundary. After an introduction to boundary integral equations, this review describes some of the methods which have been proposed for their approximate solution. It discusses, as simply as possible, some of the techniques used in their error analysis, and points to areas in which the theory is still unsatisfactory.

CONTENTS

1. Introduction

In the past decade there has been a dramatic growth of engineering interest in boundary integral or boundary element methods, witnessed by the large number of recent conference proceedings with these words in the title. At the same time, the former rivalry between advocates of BIE (boundary integral equation) and PDE (partial differential equation) approaches seems to have softened, as the relative strengths and weaknesses of each have become better understood.

Boundary integral methods may be used for interior and exterior problems, but have a special advantage for the latter. As a first introduction,

consider (without equations!) the problem of acoustic scattering from an object. Under appropriate idealization, the pressure in the region exterior to the object satisfies the wave equation, with an appropriate (typically Neumann) condition on the scattering surface, and a radiation condition at infinity. With the time variable separated, the equation becomes the Helmholtz equation. Regarded as a PDE problem, the setting is an infinite three-dimensional region. The boundary integral formulation of this problem, on the other hand, lives in a region that is only two-dimensional and finite – namely the surface of the scatterer.

We leave until the next section any serious discussion of boundary integral formulations (and refer to Colton and Kress (1983) for the specific matter of the Helmholtz equation), but some readers may find the following thought useful: If we knew the Green's function for this scattering problem, then the pressure at any point could be found by quadrature over the surface. But the true Green's function, incorporating the boundary condition on the scatterer, is even harder to find than the solution itself. The next best thing is to use the known fundamental solution, which is the Green's function for the infinite region with *no* scatterer. That incorporates the boundary condition at infinity, and solves the differential equation, but takes no account of the scatterer. To obtain a solution that satisfies the boundary conditions on the scatterer we must therefore solve for an unknown function over the surface of the scatterer. The equation to be solved is a (boundary) integral equation.

Compared to PDE formulations, those involving BIEs are usually of lower dimensionality (e.g. two-dimensional against three-dimensional in the earlier example). On the other hand BIE methods almost invariably have dense matrices, in contrast to the sparse matrices given by the PDE methods. Moreover, the matrix elements are relatively hard to compute, involving for example weakly or strongly singular kernels, perhaps (particularly in the Galerkin method) several levels of integration, and difficult geometry. Boundary integral equations rely fundamentally on the linear superposition of solutions, and therefore are happiest when the underlying differential equations are linear and homogeneous, and the material properties constant. The PDE methods in contrast, being local in character, are not so fussy about any of these matters.

Nevertheless, in the circumstances in which they are appropriate, boundary integral methods can be very useful. And their applicability can be widened by the coupling of PDE and BIE methods, using each in the regions where they are appropriate. (See, for example, Zienkiewicz *et al.* (1977), Johnson and Nedelec (1980), Costabel (1987) and recent reviews by Hsiao (1990, 1991).)

In this review our concern is with the numerical analysis of boundary integral methods and, in particular, with certain recent developments. The

review could not hope to be exhaustive, given the range and the complexity of the subject. We are helped, however, by the existence of two other recent reviews, by Wendland (1990) and Atkinson (1990). The former gives a comprehensive overview of the recent theory of the Galerkin method for BIEs via the theory of strong ellipticity for pseudodifferential operators. It consists of lecture notes for an audience with a strong PDE background; those without such a background might find the present review a useful introduction. The review by Atkinson (1990) lays particular stress on the problems involved in the implementation of three-dimensional boundary integral equations, such as the problem of evaluating the (often weakly singular) integrals over boundary elements, and iterative methods for the solution of the dense linear systems that result. We shall not consider such questions in the present review.

Nor can we do justice to the large body of recent work on (Cauchy) singular integral equations in the plane (see, for example, Prössdorf and Silbermann (1977, 1991), Prössdorf (1989)). Mixed boundary value problems will be ignored (see Wendland *et al.* (1979), Lamp *et al.* (1984), Stephan and Wendland (1985), McLean (1990)). And we will have nothing to say about another topic currently attracting considerable interest, namely nonlinear aspects of BIE (see Ruotsalainen and Wendland (1988), Ruotsalainen and Saranen (1989), Atkinson and Chandler (1990), Eggermont and Saranen (1990), Ruotsalainen (1992); and, in connection with coupling of BIEs and PDEs, Gatica and Hsiao (1989)).

Many aspects of linear integral equations relevant to BIEs and their linear approximation are discussed carefully in the recent books by Kress (1989), Hackbusch (1989), and Prössdorf and Silbermann (1991).

In the later part of the review we will give particular attention to problems in the plane, because this has been an area which has seen considerable recent activity, with many new methods proposed, some new techniques of analysis, and some attempt to tackle the challenging problems posed by corners. Perhaps some of these methods will subsequently be extended to the even more challenging three-dimensional problems.

The structure of this review is as follows. In the next Section a simple introduction is given to the BIE formulation of the problem. Sobolev spaces and the mapping properties of boundary integral operators cannot be avoided in the modern numerical analysis of BIEs. They are introduced gently in Section 3, and then, based on this knowledge, existence and uniqueness questions are considered in Section 4. Aside from their importance for the basic theory, the Fourier series techniques introduced in Section 3 will play a major role later in the paper, for both the analysis and design of approximate methods.

Of all the methods mentioned in this review, the only one which is in a reasonably satisfactory condition for a wide class of BIEs is the Galerkin

method. At heart, this is because it rests on a variational principle. A simple treatment is given in Section 5. Section 6 is devoted to the collocation method, and Section 7 to the so-called qualocation method and its discrete variants, for BIEs on plane curves. In general the analysis of these methods for smooth curves is reasonably satisfactory, but problems remain where there are corners. Section 8 summarizes some challenges for the future from corner and other problems. At the same time it discusses briefly an extreme case of a corner (the case of the logarithmic-kernel integral equation for a slit), for which a complete analysis is available. Perhaps this case may give some insight into the proper handling of general corners for this and other problems.

It might reasonably be said that in its theoretical analysis the boundary integral method is a decade or more behind the finite element method. A defence might be that the problem is genuinely harder, because of the non-local nature of integral operators. In any event, there can be no argument that there is still much to be done.

2. Boundary integral equations

The reformulation of elliptic boundary value problems as boundary integral equations has been discussed by many people, including Jaswon (1963) and Jaswon and Symm (1977) for potential theory and elastostatics, Kupradze (1965) for elasticity, and Colton and Kress (1983) for the Helmholtz equation. Clements (1981) considers general second-order elliptic problems, Hsiao and MacCamy (1973) and Hsiao (1989) concentrate on first-kind formulations, Ingham and Kelmanson (1984) consider biharmonic and singular problems, and Wendland (1990) discusses a range of examples. In addition there are many books and papers with an engineering flavour, among which we may mention Hess and Smith (1967), Brebbia *et al.* (1984), Banerjee and Watson (1986), and the introductory book by Hartmann (1989); for a more complete bibliography of engineering works see Atkinson (1990).

The classical mathematical formulations are discussed thoroughly by Mikhlin (1970). An excellent source for modern mathematical developments is the recent review of boundary integral equations by Maz'ya (1991).

Here our aim is merely to introduce some of the principal ideas in a simple setting, with no attempt at completeness or maximum generality.

2.1. Indirect methods

Consider the two-dimensional Laplace equation

$$\Delta\phi = 0, \quad t \in \Omega, \tag{2.1}$$

subject to the Dirichlet boundary conditions

$$\phi = g \quad \text{on} \quad \Gamma = \partial\Omega, \tag{2.2}$$

where Ω is a simply-connected open domain in the plane with a piecewise-smooth boundary Γ. The simplest boundary integral formulation of this problem is via the 'single-layer representation' of the potential ϕ; that is, one seeks a representation of ϕ in the form

$$\phi(t) = -\frac{1}{\pi}\int_\Gamma \log|t-s| z(s)\,\mathrm{d}l_s, \quad t \in \Omega, \tag{2.3}$$

where $|t-s|$ is the Euclidean distance between t and s, $\mathrm{d}l$ is the element of arc length, and z is an unknown function, the 'single-layer density', or 'charge density'. The motivation is easily stated: because $(2\pi)^{-1}\log|t-s|$ is the fundamental solution of the Laplace equation, (2.3) yields a solution of the Laplace equation, no matter how z is chosen; thus all that remains is to satisfy the boundary condition (2.2). Letting t approach the boundary, and assuming that the right-hand side of (2.3) is continuous onto the boundary, we obtain

$$g(t) = -\frac{1}{\pi}\int_\Gamma \log|t-s| z(s)\,\mathrm{d}l_s, \quad t \in \Gamma.$$

This is an integral equation of the first kind (which merely means that the unknown z occurs only under the integral sign). Introducing the single-layer integral operator V defined by

$$Vv(t) = -\frac{1}{\pi}\int_\Gamma \log|t-s| v(s)\,\mathrm{d}l_s, \quad t \in \Gamma, \tag{2.4}$$

we may write the integral equation as

$$Vz = g. \tag{2.5}$$

That the integral on the right-hand side of (2.3) is continuous as $t \to \Gamma$ has been shown by Gaier (1976), under the assumption that $z \in L_p(\Gamma)$ for some $p > 1$, and that the curve is piecewise smooth and has no cusps.

Next, consider the exterior problem

$$\Delta\phi = 0, \quad t \in \Omega_e, \tag{2.6}$$

where $\Omega_e = \mathbb{R}^2 \backslash \bar{\Omega}$ and Ω is defined as for (2.1). Again we assume the Dirichlet condition (2.2) on Γ, but this time we need also a regularity condition at infinity,

$$\phi \quad \text{bounded at infinity.} \tag{2.7}$$

Following Jaswon and Symm (1977), it is natural to seek a representation in the form

$$\phi(t) = -\frac{1}{\pi}\int_\Gamma \log|t-s| z(s)\,\mathrm{d}l_s + \omega, \tag{2.8}$$

where ω is a constant, and where, in order to satisfy the condition at infinity,

the side condition

$$\int_\Gamma z(s)\,\mathrm{d}l_s = 0$$

is imposed. In this case the corresponding boundary integral equation is the pair

$$Vz + \omega = g, \quad \int_\Gamma z = 0. \tag{2.9}$$

In the preceding paragraph we considered the exterior problem, but in fact there is nothing to stop us from using the same approach, of an additional unknown ω and a side condition on z, even for the interior problem (2.1), (2.2). That approach has been advocated by Hsiao and MacCamy (1973), in order to avoid the existence/uniqueness problems that can beset (2.5) (see Subsection 4.3). There is a close relationship between the two approaches: for example, if the pair ω, $z^{(1)}$ satisfies (2.9) and if $Vz^{(2)} = 1$ then it is obvious that $z = z^{(1)} + \omega z^{(2)}$ satisfies (2.5). For an elaboration of this relationship see Sloan and Spence (1988a, Section 3).

Three-dimensional interior and exterior problems for the Laplace equation with Dirichlet boundary condition (and with the regularity condition $\phi(t) \to 0$ as $|t| \to \infty$ in the exterior case) may be approached in a manner analogous to (2.3), the fundamental solution in this case being the Newtonian potential, the single-layer representation being

$$\phi(t) = \frac{1}{2\pi}\int_\Gamma \frac{1}{|t-s|} z(s)\,\mathrm{d}S_s, \quad t \in \Omega \text{ or } \Omega_e, \tag{2.10}$$

and the single-layer operator on $\Gamma = \partial\Omega$ being

$$Vz(t) = \frac{1}{2\pi}\int_\Gamma \frac{1}{|t-s|} z(s)\,\mathrm{d}S_s, \quad t \in \Gamma. \tag{2.11}$$

The resulting integral equation, for both interior and exterior problems, is (2.5).

2.2. The classical BIEs of potential theory

Returning to the two-dimensional case, the classical approach to the interior Dirichlet problem (2.1) and (2.2) is to seek a 'double-layer' representation for ϕ, i.e.

$$\begin{aligned} \phi(t) &= \frac{1}{\pi}\int_\Gamma \left(\frac{\partial}{\partial n_s}\log|t-s|\right) z(s)\,\mathrm{d}l_s \\ &= \frac{1}{\pi}\int_\Gamma \frac{n(s)\cdot(s-t)}{|t-s|^2} z(s)\,\mathrm{d}l_s, \quad t \in \Omega. \end{aligned} \tag{2.12}$$

Here the derivative is the normal derivative (with respect to s), in the direction of the outward unit normal n (i.e. the normal directed into the exterior

region Ω_e). This approach leads to a quite different kind of equation, because the double-layer operator on the right of (2.12) is generally not continuous onto Γ. The following theorem is proved by Mikhlin (1970) for the case of a Lyapunov curve, and by Hackbusch (1989) and Wendland (1990) for a curve with corners.

Theorem 1 Let Γ be piecewise Lyapunov without cusps, and let $z \in C(\Gamma)$. Then the integral

$$\frac{1}{\pi}\int_\Gamma \left(\frac{\partial}{\partial n_s}\log|t-s|\right) z(s)\,\mathrm{d}l_s, \quad t \in \mathbb{R}^2\backslash\Gamma, \tag{2.13}$$

has limiting values as t approaches Γ from Ω and Ω_e separately. If $t' \in \Gamma$ is a point at which Γ has a tangent, the limiting values as $t \to t'$ are

$$\frac{1}{\pi}\int_\Gamma \left(\frac{\partial}{\partial n_s}\log|t'-s|\right) z(s)\,\mathrm{d}l_s \pm z(t'), \tag{2.14}$$

where the upper and lower signs hold for $t \in \Omega$ and $t \in \Omega_e$ respectively.

The proof proceeds by representing $z(s)$ in (2.13) as $z(t') + (z(s) - z(t'))$, and showing that the integral corresponding to the second term is continuous onto Γ. For the first term, because $z(t')$ can be taken outside the integral, it is sufficient to prove the result for $z \equiv 1$. Briefly, for $s \in \Gamma$ and for t a fixed point in Ω, Ω_e or Γ (but not a corner point of Γ), let ρ, θ be polar coordinates of $s - t$, and let ψ be the angle between the outward normal $n(s)$ and the vector $s - t$. Then

$$\frac{\partial}{\partial n_s}\log|t-s| = \frac{n(s)\cdot(s-t)}{\rho^2} = \frac{\cos\psi}{\rho},$$

and

$$\mathrm{d}l_s = \frac{\rho\,\mathrm{d}\theta}{\cos\psi},$$

thus

$$\frac{1}{\pi}\int_\Gamma \left(\frac{\partial}{\partial n_s}\log|t-s|\right)\mathrm{d}l_s = \frac{1}{\pi}\int \mathrm{d}\theta = \frac{1}{\pi}\begin{cases} 2\pi & \text{if} \quad t \in \Omega, \\ \pi & \text{if} \quad t \in \Gamma, \\ 0 & \text{if} \quad t \in \Omega_e. \end{cases} \tag{2.15}$$

That is the integral in (2.13) has in this case the value 2 or 0 for t in Ω or Ω_e respectively, while (2.14) has the value 1 ± 1.

Let

$$\begin{aligned} Kz(t) &= \frac{1}{\pi}\int_\Gamma \left(\frac{\partial}{\partial n_s}\log|t-s|\right) z(s)\,\mathrm{d}l_s \\ &= \frac{1}{\pi}\int_\Gamma \frac{n(s)\cdot(s-t)}{|t-s|^2} z(s)\,\mathrm{d}l_s, \quad t \in \Gamma, \end{aligned} \tag{2.16}$$

be the double-layer operator on Γ. Then if t' is a point on Γ at which a tangent exists, the limiting values (2.14) become

$$Kz(t') \pm z(t'). \tag{2.17}$$

If we now return to the double-layer representation (2.12) of the interior Dirichlet problem (2.1), (2.2), we see, by taking the limit as t approaches a point on the boundary and using the theorem, that z satisfies

$$g(t) = Kz(t) + z(t), \quad t \in \Gamma, \quad t \text{ not a corner point.} \tag{2.18}$$

This is an equation of the *second* kind, in the nomenclature introduced by Fredholm. If Γ is a Lyapunov curve the kernel of the integral operator K when appropriately parametrized turns out to be weakly singular (see Mikhlin (1970)). Indeed, if Γ is a C^2 curve then the kernel is even continuous. In these cases the integral operator K is a compact operator on $C(\Gamma)$, and the classical Fredholm theory applies.

For a region with corners K is no longer compact on $C(\Gamma)$ (or indeed any other space), and the Fredholm theory is inapplicable. However, it is by now well understood that the double-layer equation (2.18) can still be a very effective tool (see, for example, Verchota (1984), Costabel (1988), Hackbusch (1989), Maz'ya (1991)). In particular, Verchota (1984) shows that the jump relations in Theorem 1 hold, almost everywhere on Γ, even for general Lipschitz curves (and hence for all piecewise Lyapunov curves without cusps), and with the density function z allowed to be merely in $L_2(\Gamma)$, provided that the double-layer operator K is defined with appropriate care. (Specifically, one need only replace 3 by 2 and 2π by π in the three-dimensional generalization (2.27) given below.) The precise nature of the operator K at a two-dimensional corner was first elucidated by Radon (1919), and further discussed by Cryer (1970); see also Atkinson and de Hoog (1984) for a study of the Dirichlet problem for a wedge.

In the same way the exterior Dirichlet problem (2.6), (2.2), (2.7) may be approached by the double-layer representation (2.12). In this case the jump relations lead to an operator equation on Γ with a different sign,

$$g(t) = Kz(t) - z(t), \quad t \in \Gamma, \quad t \text{ not a corner point.} \tag{2.19}$$

The classical approach to the Neumann problem

$$\Delta\phi = 0, \quad t \in \Omega, \quad \frac{\partial\phi}{\partial n} = h \text{ on } \Gamma, \tag{2.20}$$

or to the corresponding exterior problem satisfying also (2.7), is via the single-layer representation (2.3) or (2.10). It can be shown (Mikhlin, 1970) that for z integrable on Γ and $t \notin \Gamma$ the potential $\phi(t)$ can be differentiated

under the integral sign, giving for the two-dimensional case

$$\begin{aligned}\nabla\phi &= -\frac{1}{\pi}\int_\Gamma (\nabla_t \log|t-s|)z(s)\,\mathrm{d}l_s \\ &= -\frac{1}{\pi}\int_\Gamma \frac{t-s}{|t-s|^2}z(s)\,\mathrm{d}l_s, \quad t\notin\Gamma.\end{aligned}$$

Letting t' denote a point on Γ and $\partial\phi/\partial n$ the directional derivative in the direction of the (outward) normal at t', we have

$$\frac{\partial\phi}{\partial n} = -\frac{1}{\pi}\int_\Gamma \frac{n(t')\cdot(t-s)}{|t-s|^2}z(s)\,\mathrm{d}l_s, \quad t\notin\Gamma. \tag{2.21}$$

The normal derivative has jump discontinuities analogous to those in Theorem 1 as $t\to t'$. For the case of a Lyapunov surface the limits as $t\to t'\in\Gamma$ are (Mikhlin, 1970)

$$-K^*z(t') \pm z(t'), \tag{2.22}$$

where again the upper and lower signs hold for $t\in\Omega$ and Ω_e respectively, and

$$\begin{aligned}K^*z(t) &= \frac{1}{\pi}\int_\Gamma \left(\frac{\partial}{\partial n_t}\log|t-s|\right)z(s)\,\mathrm{d}l_s, \\ &= \frac{1}{\pi}\int_\Gamma \frac{n(t)\cdot(t-s)}{|t-s|^2}z(s)\,\mathrm{d}l_s, \quad t\in\Gamma.\end{aligned} \tag{2.23}$$

Note that the normal derivative in this case is with respect to t, whereas in the double-layer operator (2.16) it is with respect to s. In fact these operators are adjoints of each other.

It follows from this that the two-dimensional interior and exterior Neumann problems, at least for a Lyapunov curve, are characterized by the equation

$$h(t) = -K^*z(t) \pm z(t), \quad t\in\Gamma. \tag{2.24}$$

For a curve with corners comments similar to those made earlier for the double-layer equation are applicable: the same equation holds for t not a corner point (Hackbusch, 1989).

For the three-dimensional Laplace equation the double-layer representation is

$$\begin{aligned}\phi(t) &= -\frac{1}{2\pi}\int_\Gamma \left(\frac{\partial}{\partial n_s}\frac{1}{|t-s|}\right)z(s)\,\mathrm{d}S_s \\ &= \frac{1}{2\pi}\int_\Gamma \frac{n(s)\cdot(s-t)}{|t-s|^3}z(s)\,\mathrm{d}S_s, \quad t\notin\Gamma.\end{aligned} \tag{2.25}$$

Corresponding to (2.15) are the Gauss laws as a result of which jump relations analogous to those of Theorem 1 hold. This time, however, we state

the more powerful version due to Verchota (1984). (The analogous two-dimensional result also holds.) Here Ω is allowed to be an open, bounded, Lipschitz domain with connected boundary Γ, and $\Omega_e = \mathbb{R}^3 \backslash \bar{\Omega}$. Verchota (1984) shows, by making use of the celebrated Coifman *et al.* (1982) theorem, that for $z \in L_2(\Gamma)$ the limit of (2.25) as $t \to t' \in \Gamma$ exists almost everywhere on Γ, and has the value

$$Kz(t') \pm z(t'), \tag{2.26}$$

where the upper and lower signs are for the interior and exterior cases respectively, and

$$Kz(t) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\Gamma, |t-s|>\epsilon} \frac{n(s) \cdot (s-t)}{|t-s|^3} z(s) \, \mathrm{d}S_s, \quad t \in \Gamma. \tag{2.27}$$

Thus the BIEs for the interior and exterior Dirichlet problem become, as in the two-dimensional case,

$$g = Kz \pm z. \tag{2.28}$$

The operator K defined by (2.27) is a bounded operator on $L_2(\Gamma)$, about which we will have more to say when we turn to the question of existence and uniqueness. The jump relations for the normal derivatives of the single-layer potential extend to the three-dimensional situation in a similar way. Thus one obtains again BIEs of the form (2.24) for the interior and exterior Neumann problem, where for a general Lipschitz surface (Verchota, 1984)

$$K^* z(t) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\Gamma, |t-s|>\epsilon} \frac{n(t) \cdot (t-s)}{|t-s|^3} z(s) \, \mathrm{d}S_s. \tag{2.29}$$

2.3. Direct methods

The methods discussed so far are termed indirect methods, because they introduce quantities (namely, the single- or double-layer densities z on Γ) which are not part of the problem as originally formulated. Direct methods, in contrast, deal only with physically meaningful quantities, and for that reason are often favoured.

Direct methods are based on Green's theorem or its analogues. Suppose we are considering the Laplace equation (2.1) for an interior domain Ω having a smooth boundary, and suppose that $\phi \in C^2(\bar{\Omega})$ and ϕ satisfies (2.1). Then Green's theorem gives (Mikhlin, 1970, p. 224)

$$\phi(t) = \frac{1}{2\pi} \int_\Gamma \left[\left(\frac{\partial}{\partial n_s} \log |t-s| \right) \phi(s) - \log |t-s| \frac{\partial \phi(s)}{\partial n_s} \right] \mathrm{d}l_s, \quad t \in \Omega. \tag{2.30}$$

An equation on the boundary may now be obtained by letting $t \to \Gamma$ and

using the continuity properties of the single-and double-layer potentials discussed earlier: we obtain

$$\phi(t) = \frac{1}{2}(K\phi(t) + \phi(t)) + \frac{1}{2}V\frac{\partial\phi}{\partial n}(t), \quad t \in \Gamma,$$

or

$$\phi = K\phi + V\frac{\partial\phi}{\partial n}. \tag{2.31}$$

This equation is an identity, which holds whenever ϕ satisfies the Laplace equation on Ω. Thus far we have assumed stringent conditions on Γ and ϕ, but these can be relaxed significantly (see, for example, Costabel (1988)), to allow curves that are merely Lipschitz, and hence may have corners.

Now let us introduce boundary conditions. Consider first the case of the Dirichlet boundary condition (2.2). Then (2.31) gives an integral equation of the first kind for $z \equiv \partial\phi/\partial n$,

$$Vz = g - Kg. \tag{2.32}$$

Now suppose instead that the boundary condition is

$$\frac{\partial\phi}{\partial n} = \kappa\phi + h, \tag{2.33}$$

with κ a constant. Then (2.31) becomes a second kind equation for ϕ,

$$\phi = (K + \kappa V)\phi + Vh. \tag{2.34}$$

The direct method is particularly attractive in the common situation in which the boundary conditions are mixed, for example with Dirichlet boundary conditions imposed on $\Gamma_1 \subset \Gamma$, and Neumann conditions on $\Gamma\backslash\Gamma_1$. This is because starting from the identity (2.31) (which is appropriate if we assume still that the equation is the Laplace equation), we may easily develop coupled boundary integral equations for $\partial\phi/\partial n$ on Γ_1 and ϕ on $\Gamma\backslash\Gamma_1$.

3. Sobolev spaces and mappings of operators

The modern study of boundary integral equations and their numerical approximation needs some acquaintance with the mapping properties of boundary integral operators in Sobolev spaces. In the case of the Galerkin method, discussed in Section 5, information of this kind is needed for the analysis. For some of the other methods discussed in later sections a precise understanding of the operators is even more critical, in that this understanding is built into the very design of the methods. For that reason we defer the discussion of numerical methods until we have more machinery available to us.

We shall concentrate here on the two-dimensional case, with a few remarks on the three-dimensional case at the end. We make every attempt to make

the presentation as elementary as possible. A more detailed presentation from a similar point of view has been given by Kress (1989).

We shall assume for the present that Γ is a C^1 closed Jordan curve, parametrized by $t = \nu(x)$, where

$$\nu : [0,1] \to \Gamma, \quad \nu \text{ is 1-periodic}, \quad \nu \in C^1, \quad |\nu'(x)| \neq 0.$$

Any integrable function defined on Γ can be represented after this parametrization as a Fourier series,

$$v \sim \sum_{k \in \mathbb{Z}} \hat{v}(k) \mathrm{e}^{2\pi i k x},$$

where

$$\hat{v}(k) = \int_0^1 \mathrm{e}^{-2\pi i k x} v(x)\, \mathrm{d}x, \quad k \in \mathbb{Z}.$$

For any real number s we define the Sobolev norm $\|v\|_s$ of v by

$$\|v\|_s = \left(|\hat{v}(0)|^2 + \sum_{k \neq 0} |k|^{2s} |\hat{v}(k)|^2 \right)^{1/2}. \tag{3.1}$$

When $s = 0$ the norm $\|v\|_0$ is just the L_2 norm. The norm $\|v\|_s$ also has a simple enough interpretation when s is a positive integer: if we recall that the sth derivative of v has the Fourier series

$$v^{(s)} \sim \sum_{k \in \mathbb{Z}} (2\pi i k)^s \hat{v}(k) \mathrm{e}^{2\pi i k x},$$

we see that, apart from an unimportant constant factor, $\|v\|_s$ is essentially the L_2 norm of the sth derivative. (The term $|\hat{v}(0)|^2$ is included on the right of (3.1) to make this a norm, and not just a semi-norm.) Similarly, for negative integer values of s the norm is essentially the L_2 norm of the sth anti-derivative of v.

Corresponding to the norm $\|\cdot\|_s$, we introduce the Sobolev space H^s, which may be defined as the closure with respect to the norm $\|\cdot\|_s$ of the space of 1-periodic C^∞ functions. The elements of H^s are 1-periodic functions (or more generally distributions) with finite $\|\cdot\|_s$ norm. The space H^s is a Hilbert space with respect to the inner product

$$(v, w)_s = \hat{v}(0)\overline{\hat{w}(0)} + \sum_{k \neq 0} |k|^{2s} \hat{v}(k) \overline{\hat{w}(k)}. \tag{3.2}$$

An important inequality, holding for all real s and α, is

$$|(v, w)_s| \leq \|v\|_{s-\alpha} \|w\|_{s+\alpha}, \quad v \in H^{s-\alpha}, \quad w \in H^{s+\alpha}. \tag{3.3}$$

The proof is an easy application of the Cauchy–Schwarz inequality, starting

from

$$(v,w)_s = \hat{v}(0)\overline{\hat{w}(0)} + \sum_{k\neq 0} |k|^{s-\alpha}\hat{v}(k)|k|^{s+\alpha}\overline{\hat{w}(k)}.$$

We also have the stronger result

$$\|v\|_{s-\alpha} = \sup_{w\in H^{s+\alpha}} \frac{(v,w)_s}{\|w\|_{s+\alpha}}, \quad v \in H^{s-\alpha}, \tag{3.4}$$

with the supremum achieved if $\hat{w}(0) = \hat{v}(0)$, $\hat{w}(k) = |k|^{-2\alpha}\hat{v}(k)$ for $k \neq 0$. In the jargon of the trade, $H^{s-\alpha}$ and $H^{s+\alpha}$ provide a 'duality pairing' with respect to the inner product $(\cdot,\cdot)_s$.

Now that the spaces are defined, we turn to the boundary integral operators, beginning with the single-layer operator V. Writing $t = \nu(x)$, we have, from (2.4),

$$\begin{aligned} Vz(\nu(x)) &= -\frac{1}{\pi}\int_0^1 \log|\nu(x)-\nu(y)|\, z(\nu(y))\,|\nu'(y)|\,\mathrm{d}y \\ &= -2\int_0^1 \log|\nu(x)-\nu(y)|\, u(y)\,\mathrm{d}y \\ &=: Lu(x), \end{aligned} \tag{3.5}$$

where we have introduced a new unknown function

$$u(y) = \frac{1}{2\pi} z(\nu(y))\,|\nu'(y)| \tag{3.6}$$

which incorporates the Jacobian $|\nu'(y)|$ and also a convenient normalization factor.

If the curve Γ is smooth, or equivalently $\nu \in C^\infty$, the operator L defined by (3.5) behaves rather like the corresponding operator for a circle. Let A denote the operator L for the specific case of a circle of radius α. With the circle parametrized by $t = (t_1,t_2) = \alpha(\cos 2\pi x, \sin 2\pi x)$, we have explicitly

$$Au(x) = -2\int_0^1 \log|2\alpha \sin\pi(x-y)|\, u(y)\,\mathrm{d}y. \tag{3.7}$$

Then the operator L for the general curve Γ can be written as

$$L = A + B, \tag{3.8}$$

where

$$Bu(x) = -2\int_0^1 \log\left|\frac{\nu(x)-\nu(y)}{2\alpha\sin\pi(x-y)}\right| u(y)\,\mathrm{d}y. \tag{3.9}$$

For the case in which Γ is a C^∞ curve, whereas L and A have kernels which contain logarithmic singularities, the kernel of B is a C^∞ 1-periodic function of two variables. Thus for $v \in H^s$ with $s \in \mathbb{R}$ it follows that Bv is

a C^∞ function, from which we see easily that

$$B : H^s \to H^t \quad \text{for all} \quad s, t \in \mathbb{R}. \tag{3.10}$$

This fact will often allow us to treat B as a compact perturbation. (Be warned, however, that this strategy fails if Γ has corners: for then B does *not* have a smooth kernel, and the compact perturbation approach fails.)

The operator A (i.e. the single-layer operator for the case of a circle of radius α) turns out to have the following extraordinarily simple Fourier representation.

Proposition 1

$$Av(x) \sim -2\log\alpha\hat{v}(0) + \sum_{k\neq 0} \frac{1}{|k|}\hat{v}(k)\mathrm{e}^{2\pi ikx}. \tag{3.11}$$

This follows from the well known Fourier cosine series representation, valid for $x \neq 0$,

$$-\log(2|\sin\pi x|) = \sum_{k=1}^{\infty} \frac{1}{k}\cos 2\pi kx = \frac{1}{2}\sum_{k\neq 0}\frac{1}{|k|}\mathrm{e}^{2\pi ikx},$$

or equivalently

$$-2\log(2\alpha|\sin\pi x|) = -2\log\alpha + \sum_{k\neq 0}\frac{1}{|k|}\mathrm{e}^{2\pi ikx}.$$

Equation (3.11) tells us that the effect of the operator A on the kth Fourier component of v, $k \neq 0$, is to multiply that component by $1/|k|$. Recalling the definition of the Sobolev norm $\|\cdot\|_s$, it follows immediately that, for $v \in H^s$,

$$\|Av\|_{s+1} \leq c\|v\|_s, \tag{3.12}$$

and hence

$$A : H^s \to H^{s+1}. \tag{3.13}$$

Assuming for the present that Γ is a C^∞ curve, it follows that

$$L : H^s \to H^{s+1}. \tag{3.14}$$

(Throughout the paper c denotes a constant which may take different values at its different occurrences.)

The mapping property (3.14) tells us, in effect, that L is a 'once-smoothing' operator, but conveys only limited information about L. A more precise statement is that L is a 'pseudo-differential operator of order -1 and principal symbol $|\xi|^{-1}$'. That means (following Agranovich (1979)) that L can

be represented in the form

$$Lv(x) = \sum_{k \neq 0} \frac{1}{|k|} \hat{v}(k) e^{2\pi i k x} + \int_0^1 m(x,y) v(y)\, dy, \tag{3.15}$$

where m is a smooth kernel. This follows from (3.8), (3.9) and (3.11). Technically, the order is -1 because $|\xi|^{-1}$ is a positive-homogeneous function of degree -1. More general pseudo-differential operators exist, for example the principal symbol may change sign with ξ, or may depend on x. For the general form see Agranovich (1979) or Wendland (1990).

The identity operator I is a pseudo-differential operator of order 0 and principal symbol 1. So too is the operator $I + K$ arising in Section 2 from the double-layer approach to the Dirichlet problem (see (2.16)) for the case of a C^∞ curve Γ, since the double-layer operator K has in that case a C^∞ kernel.

Other pseudo-differential operators which arise in boundary integral methods are the Cauchy singular integral operator

$$Cv(t) = \frac{1}{\pi i} \int_\Gamma \frac{v(s)}{s - t}\, ds, \quad t \in \Gamma,$$

where s and t are taken to be complex numbers and the integral is to be understood in the principal-value sense, which is a pseudo-differential operator of order 0 and principal symbol sign ξ; and the normal derivative of the double-layer potential (or the 'hypersingular' operator), which is a pseudo-differential operator of order $+1$ and principal symbol $|\xi|$.

For three-dimensional surfaces $\Gamma = \partial\Omega$ the Sobolev spaces cannot be defined in quite such an elementary way, because there is no equivalent of the 1-periodic parametrization. Rather, one must use the machinery of local coordinate transformations, C^∞ cut-off functions, and Fourier transforms (see, for example, Wendland (1990)). Correspondingly, the definition of pseudo-differential operators needs to be based on Fourier transforms, rather than Fourier series. (In fact, strictly speaking this is true even in the two-dimensional case. However, the equivalence of the simpler Fourier series approach has been demonstrated by Agranovich (1979); see also Saranen and Wendland (1987) and McLean (1991).)

Nevertheless, the main results can be stated just as simply: for example, assuming that Γ is the smooth boundary of a simply connected open region, the single-layer operator defined by (2.11) is a pseudo-differential operator of order -1, while the operator $I + K$, with K the double-layer operator, is a pseudo-differential operator of order 0, and so on.

For regions with corners all of the considerations in this section require substantial modification. The Fourier series approach to the two-dimensional single-layer operator becomes less useful, because the kernel of the operator B is no longer smooth. For the case of a polygon, parametrized for

example by arc length, the single-layer potential V may still be represented in the form (3.8) where B is given by (3.9), but now the kernel of B, far from being smooth, is discontinuous at the vertices of the polygon. This defect notwithstanding, Yan and Sloan (1988, Section 5) studied the single-layer equation for a polygon in the space $H^0 = L_2$ by using the fact that the nonsmooth part of B is, in a certain precise way, not too large. However, this kind of analysis has only limited applicability to the analysis of numerical methods, because the discretized operators typically have larger norms – for example Yan (1990) in using this approach to study a collocation method for this equation was forced to restrict attention to polygons with angles no smaller than a certain minimum. A related problem is that the boundary integral operators are no longer classical pseudo-differential operators. Considerable progress has been made in the study of these operators for regions with corners and edges, see Costabel and Stephan (1985) and Costabel (1988), with, for example, Mellin transforms replacing the Fourier transforms of the classical theory. However, this is a difficult subject, into which we will not venture further.

4. Existence and uniqueness questions

4.1. Introduction

Knowledge of existence and uniqueness of the exact solution is always a precondition for a satisfactory numerical analysis. In the present context we have an added interest, in that the methods used for the exact equation often have a parallel in the analysis of approximate methods.

The classical boundary integral formulations are equations of the second kind. The analysis of these, indicated in the next subsection, uses the classical Fredholm theory in the case of reasonably smooth curves or surfaces, and more sophisticated variants when corners or edges are present.

In more recent times there has been great interest in other formulations, particularly integral equations of the first kind such as those seen already in Section 2. The extension of these to more general differential equations has been considered by Fichera (1961), and more recently by Hsiao and MacCamy (1973); see also Giroire and Nedelec (1978) and, for a review, Hsiao (1989). We consider in detail the case of the logarithmic-kernel first-kind equation in the plane, which has some interesting features, and then consider briefly more general problems. In the analysis of these more general problems the notion of strong ellipticity has come to play an important role.

4.2. Equations with second kind structure

We begin with the classical case, treated for example by Mikhlin (1970), in which Γ is taken to be a connected C^2 curve or surface. In this situation the

double-layer operator K defined by (2.16) or (2.27) is a compact operator on $L_2(\Gamma)$, as is its adjoint K^*. As the Fredholm theory is applicable, it is convenient to consider together the integral equations (2.18) for the interior Dirichlet problem,

$$z + Kz = g, \tag{4.1}$$

and (2.24) for the exterior Neumann problem

$$z + K^*z = -h, \tag{4.2}$$

since these are mutually adjoint. An argument of potential theory (see Mikhlin (1970, Chapter 18, Section 10 for the three-dimensional case, and Section 13 for the two-dimensional case)) shows that the homogeneous equation corresponding to (4.2) has only the trivial solution. The Fredholm theory (see, for example, Kress (1989)) then tells us that the same is true for the homogeneous equation corresponding to (4.1), and that both (4.1) and (4.2) have (unique) solutions $z \in L_2(\Gamma)$ for arbitrary $g \in L_2(\Gamma)$ or $h \in L_2(\Gamma)$ respectively. In other words, both $I + K$ and $I + K^*$ are boundedly invertible in $L_2(\Gamma)$.

Now consider the integral equation pair (2.19) for the exterior Dirichlet problem,

$$z - Kz = -g, \tag{4.3}$$

and (2.24) for the interior Neumann problem,

$$z - K^*z = h, \tag{4.4}$$

again mutually adjoint. This time the situation is slightly more interesting, since the Gauss laws (stated explicitly for the two-dimensional case as (2.15)) are equivalent to

$$1 - K1 = 0, \tag{4.5}$$

where 1 denotes the function on Γ whose values everywhere equal 1, so that the solution of (4.3) is not unique. It can be shown (Mikhlin 1970, Chapter 18, Sections11 and 13) that the solution space of $z - Kz = 0$ is one-dimensional, thus from the Fredholm theory the same is true of the adjoint homogeneous equation. Let f_ϵ denote the unique solution of

$$f_\epsilon - K^*f_\epsilon = 0, \quad \int_\Gamma f_\epsilon = 1. \tag{4.6}$$

Then by the Fredholm alternative (4.3) has a solution $z \in L_2(\Gamma)$ if and only if g is orthogonal to all solutions of the adjoint homogeneous equation, i.e. if and only if

$$\int_\Gamma g f_\epsilon = 0, \tag{4.7}$$

and similarly (4.4) has a solution if and only if

$$\int_\Gamma h = 0. \tag{4.8}$$

If h satisfies (4.8) and $z_0 \in L_2(\Gamma)$ is a particular solution of (4.4), it now follows that the general solution of (4.4) is $z_0 + \alpha f_\epsilon$, with α an arbitrary real number. A unique solution lying in the space

$$\overset{\circ}{L}_2(\Gamma) = \left\{ z \in L_2(\Gamma) : \int_\Gamma z = 0 \right\} \tag{4.9}$$

is then obtained by the choice $\alpha = -\int_\Gamma z_0$. In other words, $I - K^*$ is boundedly invertible in the space $\overset{\circ}{L}_2(\Gamma)$. Similarly, $I - K$ is boundedly invertible in the space

$$\tilde{L}_2(\Gamma) = \left\{ z \in L_2(\Gamma) : \int_\Gamma z f_\epsilon = 0 \right\}. \tag{4.10}$$

These arguments assume considerable regularity of Γ, but it is now known that these results hold in great generality. In particular, Verchota (1984) shows for general Lipschitz curves and surfaces that it is still true that $I + K$ is boundedly invertible in $L_2(\Gamma)$ and that $I - K^*$ is boundedly invertible in $\overset{\circ}{L}_2(\Gamma)$. It then follows by duality (Verchota, private communication) that $I + K^*$ and $I - K$ are boundedly invertible in $L_2(\Gamma)$ and $\tilde{L}_2(\Gamma)$ respectively.

The quantity f_ϵ introduced in (4.6) has an interesting interpretation. Defining a potential ψ in Ω by

$$\psi(t) = -\frac{1}{\pi} \int_\Gamma \log|t - s| f_\epsilon(s)\, \mathrm{d}l_s, \quad t \in \Omega$$

in the two-dimensional case, or

$$\psi(t) = \frac{1}{2\pi} \int_\Gamma \frac{1}{|t - s|} f_\epsilon(s)\, \mathrm{d}S_s, \quad t \in \Omega$$

in the three-dimensional case, it follows from the jump relation (2.22) for the normal derivative combined with (4.6) that

$$\frac{\partial \psi}{\partial n_-} = 0 \quad \text{on } \Gamma,$$

in which the normal derivative is the limit as Γ is approached from the interior Ω. (In the case of a Lyapunov surface this holds everywhere on Γ; for a Lipschitz surface it is valid almost everywhere – see Verchota (1984).) Since ψ is harmonic, it follows from the usual uniqueness theorem for the interior Neumann problem that ψ is constant in Ω and, hence, by the continuity of the single-layer potential,

$$V f_\epsilon = \text{constant} \quad \text{on } \Gamma \tag{4.11}$$

(almost everywhere, in the case of a Lipschitz curve). Thus f_ϵ is the 'equilibrium distribution'. In physical terms we may think of f_ϵ as the charge distribution on Γ (where the total charge is 1, since $\int_\Gamma f_\epsilon = 1$) that gives rise to a constant potential on Γ (and hence also in the interior region Ω).

We conclude this subsection with the observation that integral equation formulations are not always perfect reflections of the underlying boundary value problem. The condition (4.7), which we have seen is necessary and sufficient for the exterior Dirichlet integral equation (4.3) to have a solution, is by no means a necessary condition for the exterior Dirichlet problem itself. Mikhlin (1970, Chapter 18) discusses a modification of the equation which is solvable for every choice of the boundary-data function $g \in L_2(\Gamma)$. In a different direction, the exterior Neumann problem in two dimensions has a necessary condition, namely $\int_\Gamma h = 0$ (this is shown for example by Mikhlin (1970, Lemma 18.13.1)), which is not apparent in the integral equation formulation. Mikhlin shows (Lemma 18.13.2) that if this condition is satisfied then $\int_\Gamma z = 0$ (i.e. $I + K^*$ is boundedly invertible in $\overset{\circ}{L}_2(\Gamma)$, as well as in $L_2(\Gamma)$). The general solution of the exterior Neumann problem in two dimensions is then given by (2.8), where ω is an arbitrary constant. Because $\int_\Gamma z = 0$, this solution satisfies the boundary condition (2.7) at infinity.

4.3. The logarithmic-kernel BIE and the transfinite diameter

Before turning to more general equations, we consider the first-kind logarithmic-kernel integral equation in the plane

$$Vz(t) = -\frac{1}{\pi}\int_\Gamma \log|t-s|z(s)\,\mathrm{d}l_s = g(t), \quad t \in \Gamma, \tag{4.12}$$

which we have seen arising in Section 2 from both direct and indirect approaches to the Laplace equation with Dirichlet boundary conditions. It turns out that there is a genuine uniqueness/existence difficulty if the linear scale of the problem is inappropriate (Jaswon and Symm, 1977; Hsiao, 1986; Sloan and Spence, 1988a). Even for the case of a circle equation (4.12) may run into trouble: from (3.11), which gives the explicit Fourier representation of Vz for a circle of radius α, we see that if the radius α is 1 then z = constant implies $Vz = 0$ (since then $\log\alpha = \log 1 = 0$). Thus the solution is not unique for the case of a circle of unit radius. Moreover, for a circle of this radius it is clear from (3.11) that there is *no* solution if $g \equiv 1$.

It is well known that a similar problem arises no matter what the geometry of Γ: there is always some linear scaling of Γ for which the solution is nonunique, and no solution exists for a constant right-hand side. (Jaswon and Symm (1977) refer to a contour with this bad scaling as a 'Γ-contour'.) The essential argument depends on nothing more than the properties of the

logarithm: suppose that for a given contour Γ the equation

$$Vf_\epsilon(t) = \frac{u}{\pi}, \quad t \in \Gamma, \quad \int_\Gamma f_\epsilon = 1, \tag{4.13}$$

has a solution $f_\epsilon \in L_1(\Gamma)$ for some real number u. Then for the re-scaled contour $\Gamma' = C^{-1}\Gamma$, where

$$C = \exp(-u) \tag{4.14}$$

we find, for $t' \in \Gamma'$ and $t = Ct'$, that

$$\begin{aligned} -\frac{1}{\pi}\int_{\Gamma'} \log|t'-s'| f_\epsilon(Cs')\,\mathrm{d}l_{s'} &= -\frac{C^{-1}}{\pi}\int_\Gamma \log(C^{-1}|t-s|) f_\epsilon(s)\,\mathrm{d}l_s \\ &= C^{-1}\left(-(\log C^{-1})\frac{1}{\pi}\int_\Gamma f_\epsilon(s)\,\mathrm{d}l_s + Vf_\epsilon(t)\right) \\ &= C^{-1}\left(-\frac{u}{\pi}+\frac{u}{\pi}\right) = 0. \end{aligned}$$

Thus the logarithmic-kernel equation on the rescaled contour Γ' has a non-unique solution.

The number $C = C_\Gamma$ is a length associated with the contour Γ: it is easily seen that $C_{a\Gamma} = aC_\Gamma$. It is called the 'transfinite diameter' or 'logarithmic capacity' of Γ. (We prefer the former name, as it reminds us that C_Γ scales as a length.) In the preceding paragraph the rescaled curve Γ' has a transfinite diameter equal to 1. It also has a nontrivial solution of the homogeneous logarithmic-kernel equation. This observation should persuade us that for this equation contours of transfinite diameter 1 are to be avoided.

The argument in the preceding paragraphs depends on the existence of a solution of (4.13). Fortunately, it can be shown that a solution exists under very general conditions. For example, Hille (1962, p.280), assuming only that Γ is a closed bounded set in the plane, gives a variational definition of $u = u_\Gamma$ (the 'Robin constant'), as

$$u_\Gamma = \inf\left(-\int_\Gamma\int_\Gamma \log|t-s|\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(s)\right), \tag{4.15}$$

where the infimum is over all normalized positive measures μ defined on Γ (i.e. $\mu \geq 0$, $\int_\Gamma \mathrm{d}\mu = 1$). The transfinite diameter C_Γ is then defined by (4.14). (Actually Hille gives independent definitions of transfinite diameter and logarithmic capacity, but shows them to be equivalent, in Theorem 16.4.4.) He shows moreover (in Theorem 16.4.3) that there exists a unique normalized positive measure μ_ϵ which achieves the infimum in (4.15), and that, except possibly for t in a set of transfinite diameter zero, one has (Hille, 1962, Theorem 16.4.8)

$$-\int_\Gamma \log|t-s|\,\mathrm{d}\mu_\epsilon(s) = u_\Gamma, \quad t \in \Gamma. \tag{4.16}$$

Thus a solution of (4.13) always exists in the sense of a measure.

Some useful properties of the transfinite diameter established in Hille (1962, Chapter 16) are:

1 the transfinite diameter of Γ does not exceed its Euclidean diameter;
2 if Γ lies inside Γ', then $C_\Gamma \leq C_{\Gamma'}$;
3 the transfinite diameter of a circle of radius α is α; and
4 the transfinite diameter of an interval of length l is $l/4$.

For our present purposes it is sufficient to restrict Γ to be the union of a finite number of C^2 arcs, having only a finite number of points of intersection. Note that this is both more restrictive and less restrictive than we have assumed in preceding sections: more restrictive because we do not allow general Lipschitz curves; less restrictive in that open arcs, cusps and multiple points of intersection are allowed. Under these conditions it follows from classical arguments that $\mathrm{d}\mu_\epsilon$ has the form of a classical distribution $f_\epsilon \, \mathrm{d}l$, where $f_\epsilon \in L_1(\Gamma)$. Indeed, much more can be said about f_ϵ. Let

$$\begin{aligned} \phi_\epsilon(t) &= -\frac{1}{\pi}\int_\Gamma \log|t-s| \, \mathrm{d}\mu_\epsilon(s) \\ &= -\frac{1}{\pi}\int_\Gamma \log|t-s| f_\epsilon(s) \, \mathrm{d}l_s, \quad t \in \mathbb{R}^2, \end{aligned}$$

be the potential corresponding to the equilibrium distribution. By standard arguments (e.g. Gaier (1976)) ϕ_ϵ is continuous on $\mathbb{R}^2$, except possibly at ends of arcs, cusps and points of intersection, so that

$$\phi_\epsilon(t) \to V f_\epsilon(t) = \frac{u_\Gamma}{\pi} \quad \text{as } t \to \Gamma.$$

With the same exceptions the jump relations (2.22) for the normal derivative hold in a pointwise sense on Γ, from which it follows that

$$f_\epsilon = -\frac{1}{2}\left(\frac{\partial \phi_\epsilon}{\partial n_+} - \frac{\partial \phi_\epsilon}{\partial n_-}\right), \tag{4.17}$$

where the normal derivatives are the limits as Γ is approached from the positive and negative sides (with respect to a normal with arbitrary but fixed sense). The known regularity property of the solutions of the Laplace equation now allows us to infer that f_ϵ is continuous on Γ except at points of intersection, cusps or ends. On the other hand f_ϵ is singular at a free end: for example, for an arc lying on the positive x-axis, with one end at the origin, in a neighbourhood of the origin we have

$$\phi_\epsilon(r\cos\theta, r\sin\theta) = \frac{u_\Gamma}{\pi} + cr^{1/2}\sin\frac{\theta}{2} + c'r\sin\theta + \mathcal{O}(r^{3/2}), \quad 0 < \theta < 2\pi,$$

from which it follows, using (4.17), that

$$f_\epsilon(x) = cx^{-1/2} + \mathcal{O}(x^{1/2}) \tag{4.18}$$

in a neighbourhood of the end. Similar but weaker singularities occur at corners; for details see Sloan and Spence (1988b). Note that $f_\epsilon \notin L_2(\Gamma)$ if there is a free end. On the other hand if Γ is the piecewise-smooth-without-cusps boundary of an open domain Ω, then $f_\epsilon \in L_2(\Gamma)$, and is just the function we met in the previous subsection as the solution of the second-kind integral equation (4.6) (since (4.11) is equivalent to (4.13)).

For $C_\Gamma \neq 1$ the solution of (4.12), if it exists, is unique. One way to show this is to decompose $z \in L_1(\Gamma)$ in the form

$$z = \alpha f_\epsilon + z_0, \tag{4.19}$$

with

$$z_0 \in \mathring{L}_1(\Gamma) = \left\{ w \in L_1(\Gamma) : \int_\Gamma w = 0 \right\}. \tag{4.20}$$

Since this decomposition is always possible with a uniquely determined α, namely

$$\alpha = \int_\Gamma z,$$

the representation (4.19) corresponds to a direct sum decomposition of $L_1(\Gamma)$,

$$L_1(\Gamma) = \{\alpha f_\epsilon : \alpha \in \mathbb{R}\} \oplus \mathring{L}_1(\Gamma). \tag{4.21}$$

Corresponding to the representation (4.19) we have

$$\begin{aligned} Vz &= \alpha V f_\epsilon + V z_0 \\ &= \frac{u}{\pi} \alpha 1 + V z_0. \end{aligned} \tag{4.22}$$

By a change in the order of integration (using Fubini's theorem), we see that

$$\int_\Gamma (V z_0) f_\epsilon = \int_\Gamma z_0 (V f_\epsilon) = \frac{u}{\pi} \int_\Gamma z_0 = 0, \tag{4.23}$$

thus $V : \mathring{L}_1 \to \tilde{L}_1$, where

$$\tilde{L}_1(\Gamma) = \left\{ w \in L_1(\Gamma) : \int_\Gamma w f_\epsilon = 0 \right\}. \tag{4.24}$$

Thus (4.19) and (4.22) correspond to a direct sum representation of V, namely

$$V = V_\epsilon \oplus \mathring{V}, \tag{4.25}$$

where

$$V_\epsilon : \{\alpha f_\epsilon : \alpha \in \mathbb{R}\} \to \{\alpha 1 : \alpha \in \mathbb{R}\} \tag{4.26}$$

with $V_\epsilon f_\epsilon = V f_\epsilon = (u/\pi)1$, and

$$\mathring{V} : \mathring{L}_1(\Gamma) \to \tilde{L}_1(\Gamma). \tag{4.27}$$

The operator V_ϵ, having one-dimensional domain and co-domain, is certainly one-to-one if $u \neq 0$ (and hence if $C_\Gamma \neq 1$); and Doob (1984) shows that the operator $\overset{\circ}{V}$ is positive definite (and hence one-to-one), in the sense that

$$-\int_\Gamma \int_\Gamma \log|t-s| z_0(s)\,\mathrm{d}l_s z_0(t)\,\mathrm{d}l_t \geq 0 \tag{4.28}$$

for all $z_0 \in \overset{\circ}{L}_1$. (In fact Doob's result holds for the much larger class of signed measures with mean zero.) Thus uniqueness is proved.

When does a solution of (4.12) exist? If Γ is a C^∞ curve then the existence can be discussed in terms of the Sobolev spaces introduced in Section 3. As in (3.5) we define $Vz(\nu(x)) = Lu(x)$, and as in (3.8) we write L as

$$L = A + B, \tag{4.29}$$

where A is the single-layer operator for the case of a circle of radius α. It is convenient to choose $\alpha = \mathrm{e}^{-1/2}$, because then we see from (3.11) that A has the especially simple Fourier series representation

$$Av(x) \sim \hat{v}(0) + \sum_{k\neq 0} \frac{1}{|k|}\hat{v}(k)\mathrm{e}^{2\pi i k x}. \tag{4.30}$$

From this it follows that A is an invertible operator from H^s onto H^{s+1} for arbitrary $s \in \mathcal{R}$, that is

$$A : H^s \to H^{s+1}, \quad A^{-1} : H^{s+1} \to H^s, \tag{4.31}$$

and, moreover, from the definition (3.1) of the Sobolev norms A is isometric:

$$\|Av\|_{s+1} = \|v\|_s. \tag{4.32}$$

Since A is invertible we may write (4.29) as

$$L = A(I + K), \tag{4.33}$$

where

$$K = A^{-1}B. \tag{4.34}$$

Now from (3.10) it follows that

$$K : H^s \to H^t \quad \text{for all } s, t \in \mathbb{R},$$

thus K is a compact operator on H^s. If we assume that $C_\Gamma \neq 1$ then, as discussed earlier, L is a one-to-one operator, thus, from (4.33), so too is $I+K$. It now follows from the Fredholm alternative that $I+K$ is boundedly invertible on H^s or equivalently,

$$I + K : H^s \to H^s, \quad \text{1–1 and onto.} \tag{4.35}$$

The final conclusion is that if $C_\Gamma \neq 1$ the operator L behaves just like the

operator for the case of a circle, in that

$$L : H^s \to H^{s+1}, \quad \text{1–1 and onto.} \tag{4.36}$$

Thus the equation $Lu = f$ has a solution $u \in H^s$ for arbitrary $f \in H^{s+1}$.

For Γ a closed Lipschitz curve Verchota (1984) shows, for $C_\Gamma \neq 1$, that (4.12) has a solution $z \in L_2(\Gamma)$ for arbitrary $g \in L_2'(\Gamma) := \{w \in L_2(\Gamma) : w' \in L_2(\Gamma)\}$, w' being the (tangential) derivative on Γ.

Before leaving the single-layer equation, it should be said that for the three-dimensional first-kind boundary integral equation $Vz = g$, with V given by (2.11), no uniqueness difficulty arises; there is no scaling for which the homogeneous equation has a nontrivial solution, and in fact V, for any scaling of Γ, is a positive definite operator. The difficulties that arise with (4.12) may be thought of as an idiosyncracy of two dimensions.

4.4. More general equations – strong ellipticity

Many of the boundary integral equations that arise in practice are 'strongly elliptic' and hence 'coercive' with respect to an appropriate Hilbert space. We shall see that this not only provides a simple way of establishing the existence and uniqueness of the exact solution (in an appropriate weak sense), but also gives a very satisfactory framework for analysing the Galerkin method (see Section 5).

For simplicity we restrict ourselves to boundary integral equations which can be written as single equations of the form

$$Lu = f. \tag{4.37}$$

In the two-dimensional case it is convenient to assume that the boundary curve Γ has already been parametrized in the manner of (3.5), so that u and f are 1-periodic functions. In the three-dimensional case u and f are functions on Γ. By restricting ourselves to equations of the form (4.37) we are excluding systems of equations, and also equations such as (2.9), in which there is a scalar unknown in addition to the unknown function u. For generalizations see Stephan and Wendland (1976) and Wendland (1983, 1985, 1987).

We shall say that (4.37) has a 'weak' solution u if, for all χ in an appropriate space,

$$(Lu, \chi)_0 = (f, \chi)_0, \tag{4.38}$$

where

$$(v, w)_0 = \int_0^1 v\overline{w} \quad \text{or} \quad \int_\Gamma v\overline{w} \tag{4.39}$$

in the two-dimensional or three-dimensional case respectively. In one important circumstance the existence of a weak solution is guaranteed: if for some

Hilbert space H the bilinear form $(L\phi, \psi)_0$ is both bounded and coercive, i.e. if for some positive constants D and ν

$$|(L\phi, \psi)_0| \leq D\|\phi\|_H \|\psi\|_H \quad \forall \phi, \psi \in H, \tag{4.40}$$

and

$$\mathrm{Re}\,(L\phi, \phi)_0 \geq \nu \|\phi\|_H^2 \quad \forall \phi \in H, \tag{4.41}$$

then the Lax–Milgram theorem (Gilbarg and Trudinger, 1983, Theorem 5.8; Ciarlet, 1978) is applicable:

Theorem 2 (Lax–Milgram) Assume that $a(\phi, \psi)$ is a bounded, coercive bilinear form on a Hilbert space H, and that F is a bounded linear functional on H. Then there exists a unique $u \in H$ such that

$$a(u, \chi) = F(\chi) \quad \forall \chi \in H.$$

It follows that $Lu = f$ has a weak solution $u \in H$ for each f for which $(f, \cdot)_0$ is a bounded linear functional on H.

A simple example is provided by the single-layer equation for a circle of radius $\alpha < 1$, already discussed in the preceding subsection. From (3.11) and (3.1) we have in this case

$$\begin{aligned} (L\phi, \psi)_0 &= (A\phi, \psi)_0 = -2\log\alpha \hat{\phi}(0)\overline{\hat{\psi}(0)} + \sum_{k \neq 0} \frac{1}{|k|} \hat{\phi}(k)\overline{\hat{\psi}(k)} \\ &\leq \max(-2\log\alpha, 1)\|\phi\|_{-1/2}\|\psi\|_{-1/2} \end{aligned} \tag{4.42}$$

and

$$(L\phi, \phi)_0 \geq \min(-2\log\alpha, 1)\|\phi\|_{-1/2}^2, \tag{4.43}$$

so that $(L\phi, \phi)_0$ is bounded and coercive with respect to the Sobolev space $H^{-1/2}$. Thus the logarithmic-kernel equation $Lu = f$ for a circle of radius $\alpha < 1$ has a weak solution $u \in H^{-1/2}$ for each $f \in H^{1/2}$. (Recall that H^s and H^{-s} are a dual pair with respect to the L_2 inner product $(\cdot, \cdot)_0$ – see (3.3), (3.4).) This is consistent with the previously established result (4.31).

How can we establish, for more general boundary integral operators, that the conditions (4.40) and (4.41) are satisfied for some Hilbert space H? For boundary integral operators on smooth closed curves or surfaces, the theory of pseudo-differential operators, already discussed briefly in Section 3, can be used to good effect. This theory, and its application to the Galerkin method, has been discussed with admirable thoroughness in a number of places, for example Stephan and Wendland (1976), Wendland (1983, 1987, 1990) and Hsiao and Wendland (1981). Here we content ourselves with a brief look at a two-dimensional case.

Suppose that L is an operator on 1-periodic functions defined by

$$Lv(x) = \sum_{k \neq 0} a(x,k)\hat{v}(k)\mathrm{e}^{2\pi i k x} + \int_0^1 m(x,y)v(y)\,\mathrm{d}y, \quad x \in [0,1], \tag{4.44}$$

where $m \in C^\infty([0,1]\times[0,1])$, and $a(x,\xi)$ is a 1-periodic C^∞ function of x for each $\xi \neq 0$, and for some $\beta > 0$ and each $x \in \mathbb{R}$ is a positive-homogeneous function of degree β in ξ. Then L is a pseudo-differential operator of order β and principal symbol $a(x,\xi)$. (The logarithmic-kernel operator defined by (3.8)–(3.11) is a pseudo-differential operator of order -1 and principal symbol $|\xi|^{-1}$.) A pseudo-differential operator of order β is (Hörmander, 1965; Wendland, 1987) a continuous operator from H^σ to $H^{\sigma-\beta}$ for all $\sigma \in \mathbb{R}$. In particular, therefore, L is a continuous operator from $H^{\beta/2}$ to $H^{-\beta/2}$. From this and (3.3) it follows that

$$|(Lv,w)_0| \leq \|Lv\|_{-\beta/2}\|w\|_{\beta/2} \leq c\|v\|_{\beta/2}\|w\|_{\beta/2},$$

so that (4.40) is satisfied with $H = H^{\beta/2}$.

Now suppose, in addition, that the principal symbol $a(x,\xi)$ is 'strongly elliptic', that is to say that for some $\mu > 0$

$$\mathrm{Re}\, a(x,\pm 1) \geq \mu \quad \forall x \in [0,1]. \tag{4.45}$$

Then it is known (Kohn and Nirenberg, 1965, p.283) that, for any $\varepsilon > 0$, $L = L_0 + M(\varepsilon)$, where L_0 is coercive with respect to $H^{\beta/2}$,

$$\mathrm{Re}(L_0\phi,\phi)_0 \geq (\mu - \varepsilon)\|\phi\|^2_{\beta/2} \quad \forall \phi \in H, \tag{4.46}$$

and $M = M(\varepsilon)$ is a compact operator from $H^{\beta/2}$ to $H^{-\beta/2}$. The addition of the compact term M leaves the conclusion of the Lax–Milgram theorem unaltered, provided that L remains one-to-one (Hildebrandt and Wienholtz (1964), Remark 3). Thus it follows in the strongly elliptic case that the equation $Lu = f$ has a weak solution $u \in H^{\beta/2}$ for each $f \in H^{-\beta/2}$.

It should be noted that the pseudo-differential operator arguments need serious modification as soon as corners or edges appear (see, for example, Costabel and Stephan (1985)).

A different and in some ways more versatile approach is to found the strong ellipticity theory for boundary integral operators on the well studied strong ellipticity properties of the associated elliptic PDEs. For further details, see Costabel and Wendland (1986). This approach has the advantage that it remains available even when corners are present, and indeed even for general Lipschitz curves (Costabel, 1988).

5. The Galerkin method

Most theoretical treatments of the boundary element method give great attention to the Galerkin method, a method originated in the context of the

differential equations of elasticity by a Russian engineer (Galerkin, 1915). The present treatment will be briefer, not least because of the very complete discussions that exist elsewhere (see, for example, Stephan and Wendland (1976), Hsiao and Wendland (1977, 1981), Wendland (1983, 1987, 1990), Rannacher and Wendland (1985, 1988)). However, the example of the two-dimensional logarithmic-kernel integral equation will be worked out in some detail.

Let us assume, as in (4.37), that the problem is expressible as a single equation of the form

$$Lu = f, \tag{5.1}$$

with u and f 1-periodic in the two-dimensional case, and functions on Γ in the three-dimensional case.

Let S_h be a finite-dimensional space within which the approximate solution is to be sought. Typically, S_h is defined by partitioning Γ into a finite number of pieces with simple geometry (e.g. plane or curved triangles) and maximum diameter h, on each of which the restriction of S_h is a piecewise polynomial space with respect to an appropriate local parametrization (e.g. one in which the element boundary is a triangle). Continuity conditions across elements may or may not be imposed, depending on the circumstances, one important constraint being $S_h \subset H$, where H is the space in (4.40–1). (For further details see, for example, Brebbia *et al.* (1984).) Then the Galerkin method is: find $u_h \in S_h$ such that

$$(Lu_h, \chi)_0 = (f, \chi)_0 \quad \forall\, \chi \in S_h, \tag{5.2}$$

where the inner product is defined by (4.39).

While the Galerkin method is the theorist's favourite, it is in truth not easy to implement. Let $\{\phi_1, \ldots, \phi_N\}$ be a basis for S_h. Then we may write

$$u_h = \sum_{j=1}^{N} a_j \phi_j, \tag{5.3}$$

and the equations to be solved in practice are

$$\sum_{j=1}^{N} (L\phi_j, \phi_k)_0 a_j = (f, \phi_k)_0, \quad k = 1, \ldots, N, \tag{5.4}$$

in which each matrix element on the left, even in the two-dimensional case, is a two-dimensional integral – one integral for the integral operator, and one for the inner product. In the three-dimensional case four levels of integration are needed for each matrix element. And the difficulty is compounded by the fact that the matrix in the boundary element method is invariably dense.

The error analysis for the Galerkin method rests on the variational formulation of the exact problem given in the preceding section. Suppose that

the bilinear form $(L\phi, \psi)_0$ satisfies the boundedness and coercivity properties (4.40) and (4.41) for some Hilbert space H, and that $u \in H$ is the weak solution of the exact equation (5.1). Then, provided only that $S_h \subset H$, Céa's lemma (Ciarlet, 1978) ensures that the Galerkin equation (5.2) has a unique solution $u_h \in S_h$, whose error in the $\|\cdot\|_H$ norm is within a constant factor of the error of best approximation by an element of S_h:

Theorem 3 (Céa's lemma) Assume that the bilinear form $(L\phi, \psi)_0$ satisfies (4.40) and (4.41), and that $Lu = f$ has the weak solution $u \in H$. Assume also that S_h is a finite-dimensional subspace of H. Then the Galerkin approximation (5.2) has a unique solution $u_h \in S_h$, which satisfies

$$\|u_h - u\|_H \leq \frac{D}{\nu} \inf_{v_h \in S_h} \|v_h - u\|_H. \tag{5.5}$$

Proof. The existence and uniqueness of u_h follows from the Lax–Milgram theorem applied to S_h as a subspace of H. Then (4.40), (4.41) and (5.2) give, for arbitrary $v_h \in S_h$,

$$\begin{aligned} \nu\|u_h - u\|_H^2 &\leq |(L(u_h - u), u_h - u)_0| = |(L(u_h - u), v_h - u)_0| \\ &\leq D\|u_h - u\|_H \|v_h - u\|_H, \end{aligned}$$

from which the result follows. □

The result (5.5) has the nice property of reducing the Galerkin error estimation in the 'natural' or 'energy' norm $\|\cdot\|_H$ to a problem of approximation.

Now let us be more explicit, and assume that Γ is a smooth curve in the two-dimensional case, or a smooth surface in the three-dimensional case, and that L is a strongly elliptic pseudo-differential operator of order β. Then, as noted in the preceding subsection, $L = L_0 + M$, where L_0 is bounded and coercive with respect to the space $H^{\beta/2}$, and M is a compact operator from $H^{\beta/2}$ to $H^{-\beta/2}$. It can be shown (Hildebrandt and Wienholtz, 1964) that the addition of the compact term leaves the essential conclusion of Céa's lemma unaltered, provided L remains one-to-one: specifically, it follows that $h_0 > 0$ exists such that $u_h \in S_h$ exists for $h < h_0$, and satisfies

$$\|u_h - u\|_{\beta/2} \leq c \inf_{v_h \in S_h} \|v_h - u\|_{\beta/2}, \tag{5.6}$$

for some constant $c > 0$.

We now specialize further to the case of the logarithmic-kernel integral equation. In the following example we indicate 'power of h' results for the Galerkin error not only in the energy norm, but also in a range of other Sobolev norms. We shall also return to the same example later, in other sections, to illustrate other numerical methods.

Example. Let Γ be a smooth curve with C_Γ (the transfinite diameter) not equal to 1, and let L be the logarithmic-kernel operator on [0,1] defined

by (3.5). Let S_h be the space of 1-periodic smoothest splines of order r (i.e. degree $\leq r-1$), where $r \geq 1$, on a partition

$$\Pi_h : 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1, \tag{5.7}$$

with

$$h_k = x_{k+1} - x_k, \quad k = 0, \ldots, N-1, \tag{5.8}$$

and

$$h = \max h_k.$$

That is, $v \in S_h$ satisfies $v \in C^{r-2}(\mathbb{R})$, and $v|_{(x_k, x_{k+1})} \in \mathbb{P}_{r-1}$. Assume, moreover, that $h \to 0$. Then it is known that for $-\infty < t \leq s \leq r$ and $t < r - \frac{1}{2}$, there exists a constant c depending only on t and s such that

$$\inf_{v_h \in S_h} \|v_h - u\|_t \leq ch^{s-t}\|u\|_s \quad \text{if } u \in H^s. \tag{5.9}$$

(For a discussion see, for example, Arnold and Wendland (1983).) In particular, therefore, it follows from (5.6) that the Galerkin error estimate in the natural norm is

$$\|u_h - u\|_{-1/2} \leq ch^{r+1/2}\|u\|_r \quad \text{if } u \in H^r. \tag{5.10}$$

Error estimates in 'lower' norms can now be deduced by a duality argument ('Nitsche's trick'), apparently first used for BIEs by Hsiao and Wendland (1981). We illustrate this for the most extreme case, namely the $\|\cdot\|_{-r-1}$ norm, in which the maximum order of convergence can be doubled over that obtained in (5.10) for the natural norm. For simplicity, we assume here that Γ is a circle of radius $\mathrm{e}^{-1/2}$, so that $L = A$, in the language of (4.29) and (4.30). Using (3.4), (3.2) and (4.30), we have

$$\begin{aligned}
\|u_h - u\|_{-r-1} &= \sup_{v \in H^r} \frac{(u_h - u, v)_{-1/2}}{\|v\|_r} = \sup_{v \in H^r} \frac{(A(u_h - u), v)_0}{\|v\|_r} \\
&= \sup_{v \in H^r} \frac{(A(u_h - u), v - v_h)_0}{\|v\|_r} \\
&\leq \sup_{v \in H^r} \frac{\|u_h - u\|_{-1/2}\|v - v_h\|_{-1/2}}{\|v\|_r},
\end{aligned}$$

where v_h is an arbitrary element of S_h, which makes its appearance at the second-last step because we have used again the Galerkin equation (5.2). Thus, by (5.10) and another application of the approximation theory result (5.9) we obtain

$$\begin{aligned}
\|u_h - u\|_{-r-1} &\leq \|u_h - u\|_{-1/2} \sup_{v \in H^r} \frac{ch^{r+1/2}\|v\|_r}{\|v\|_r} \\
&\leq ch^{2r+1}\|u\|_r.
\end{aligned} \tag{5.11}$$

Finally, error estimates in higher Sobolev norms than the $\|\cdot\|_{-1/2}$ norm may be established if the sequence of meshes is quasi-uniform; i.e. if there exists $c > 0$ such that

$$h_k \geq ch \quad \forall\, k,$$

with c independent of h. From this follows the inverse estimate (see, for example, Arnold and Wendland (1983)), for $\tau \leq \sigma < r - \frac{1}{2}$,

$$\|v\|_\sigma \leq ch^{\tau-\sigma}\|v\|_\tau \quad \text{for } v \in S_h. \tag{5.12}$$

We also make use of the fact (again see Arnold and Wendland (1983) for a discussion) that for given $u \in H^s$ there exists $\psi_h \in S_h$, *independent of* t, such that for $t \leq s \leq r$ and $t < r - \frac{1}{2}$

$$\|u - \psi_h\|_t \leq ch^{s-t}\|u\|_s \quad \text{for } u \in H^s,$$

with c independent of u. Then for $-\frac{1}{2} \leq t < r - \frac{1}{2}$ it follows that

$$\begin{aligned}
\|u_h - u\|_t &\leq \|u_h - \psi_h\|_t + \|\psi_h - u\|_t \\
&\leq ch^{-r-1-t}\|u_h - \psi_h\|_{-r-1} + \|\psi_h - u\|_t \\
&\leq ch^{-r-1-t}\|u_h - u\|_{-r-1} + ch^{-r-1-t}\|\psi_h - u\|_{-r-1} + \|\psi_h - u\|_t \\
&\leq ch^{r-t}\|u\|_r. \qquad (5.13)
\end{aligned}$$

Here we have assumed, for simplicity, that $u \in H^r$. Results for u with lesser smoothness, and correspondingly fewer powers of h, are easily written down.

The highest-order convergence in this example – of order $\mathcal{O}(h^{2r+1})$ – is obtained in the $\|\cdot\|_{-r-1}$ norm. At first sight it may not be clear why 'negative norm' results of this kind are of interest, given that they cannot be observed directly. The answer is that we do indeed see the benefit if we are interested finally not in u, but rather in an inner product $(u, w)_0$, where w is a reasonably smooth function. For from (3.3) we have

$$|(u_h, w)_0 - (u, w)_0| = |(u_h - u, w)_0| \leq \|u_h - u\|_{-r-1}\|w\|_{r+1}, \tag{5.14}$$

so that the $\mathcal{O}(h^{2r+1})$ order of convergence in the example is observable if $w \in H^{r+1}$. As a specific example of such an inner product, suppose that in the case of the logarithmic-kernel integral equation we are interested in computing, in the context of the indirect method for the interior Dirichlet problem for the Laplace equation, the potential $\phi(t)$ given by (2.3) at a point $t \notin \Gamma$. Then from (2.3) and (3.6)

$$\begin{aligned}
\phi(t) &= -\frac{1}{\pi}\int_\Gamma \log|t - s|z(s)\,\mathrm{d}l_s \\
&= -2\int_0^1 \log|t - \nu(y)|u(y)\,\mathrm{d}y \\
&= (u, w_t)_0, \qquad (5.15)
\end{aligned}$$

where w_t is the C^∞ function defined by

$$w_t(x) = -2\log|t - \nu(x)|, \quad t \notin \Gamma, \ x \in \mathbb{R}. \tag{5.16}$$

Thus if

$$\phi_h(t) = -2\int_0^1 \log|t - \nu(y)| u_h(y)\,dy \tag{5.17}$$

then

$$\phi_h(t) - \phi(t) = (u_h, w_t)_0 - (u, w_t)_0, \tag{5.18}$$

and (5.14) applies.

Many modifications of the Galerkin method have been proposed for the case of smooth plane curves. Arnold (1983) has shown that the order of negative-norm convergence can be made arbitrarily large by the use of an unsymmetric (or Petrov–Galerkin) approximation, in which the trial space remains a spline space but the test space is a space of trigonometric polynomials. McLean (1986) obtains exponential rates of convergence in stronger norms by the use of trigonometric polynomials for both test and trial spaces, if the exact solution is smooth. Atkinson (1988) obtains a similar rate of convergence with a fully discrete version of the Galerkin method with trigonometric polynomials.

The Galerkin-collocation method (Hsiao *et al.* 1980, 1984), as the name suggests, has some relation to both the Galerkin and collocation methods. In this method the logarithmic-kernel integral equation for a smooth curve, in the modified form (2.9) as advocated by Hsiao and MacCamy (1973), is handled by decomposing the operator into two parts: a principal part, which is a convolution operator, treated by a Galerkin method; and a second part, which is an integral operator with a smooth kernel, treated by a discrete approximation. (The separation into the two terms is similar to that in (3.8) and (3.9), but is different in detail.) The analysis exploits the close relation to the Galerkin method, yet the implementation is much less laborious, since the matrix corresponding to the principal part is a Toeplitz matrix, and so is representable as a vector; and moreover it is independent of the particular curve, and so can be computed once and for all. An extensive discussion of applications is given in Hsiao *et al.* (1984).

Before leaving the Galerkin method, we may mention that Rannacher and Wendland (1985,1988), by the clever use of weighted Sobolev norms, have established *uniform* error estimates for the Galerkin approximation for the single-layer equation on closed curves and surfaces.

6. The collocation method

For the solution of boundary integral equations in practice the collocation method is generally the method of choice, because it is so much easier to

implement than the Galerkin method. It does, however, have disadvantages: a theoretical analysis is available only in special cases (see later); the convergence rate in negative norms is often inferior; and the matrix is generally not symmetric even if the operator is self-adjoint.

Let us assume, as in the discussion of the Galerkin method, that the equation to be solved is a single equation of the form

$$Lu = f, \tag{6.1}$$

where u and f are either 1-periodic functions on $\mathbb{R}$, or are functions on Γ in the three-dimensional case. Let S_h be the finite-dimensional space within which the approximation is to be sought. In the collocation method one chooses also a set of 'collocation points' $t_1, \ldots, t_N$, where $N = N_h$ is the dimension of S_h. Then the collocation method is: find $u_h \in S_h$ such that

$$Lu_h(t_k) = f(t_k), \quad k = 1, \ldots, N. \tag{6.2}$$

Needless to say, the choice of the collocation points is a very important question, one to which we shall return.

Letting $\{\phi_1, \ldots, \phi_N\}$ be a basis for S_h, and writing u_h in the form (5.3), the equations to be solved in practice are

$$\sum_{j=1}^{N} L\phi_j(t_k)a_j = f(t_k), \quad k = 1, \ldots, N. \tag{6.3}$$

Clearly, the labour involved in setting up the matrix is much less than in the Galerkin method: in the two-dimensional case each matrix element requires just one integration.

Theoretical analyses of the collocation method are available in a variety of situations. For the double-layer equation (2.18) on smooth curves or surfaces, the standard analysis for Fredholm integral equations of the second kind (see, for example, Atkinson (1976) or Baker (1977)) is available. For regions with corners we saw in Section 2 that the double-layer integral operator is no longer compact, so that the standard theory is not applicable, but at least in the plane case considerable progress has nevertheless been made (see, for example, Atkinson and de Hoog (1984), Chandler and Graham (1988), Elschner (1988)). In the latter papers piecewise-polynomial collocation is shown to be stable and of optimal order, provided the approximating space is suitably modified near the corner, and the mesh is appropriately 'graded'. (The mesh is graded at a corner z, with grading parameter $q > 0$, if the points of the partition satisfy $|x_k - z| = ck^q, k = 0, 1, \ldots,$ near z).

For more general boundary integral equations on smooth plane curves an important contribution to the theoretical study of the collocation method is that of Arnold and Wendland (1983). In that paper certain collocation methods are shown to be equivalent, after integration by parts, to Galerkin

methods with nonstandard inner products. As a consequence stability and convergence are established, with almost no restrictions on the mesh. The principal limitation is that the analysis is restricted to smoothest splines of even order or odd degree (for example, continuous piecewise-linear functions), and to the particular case of collocation at the knots.

For the case of the logarithmic-kernel equation $Lu = f$ with L defined by (3.5), and for smoothest splines of (even) order r on the partition (5.7), the convergence result of highest order obtained by Arnold and Wendland (1983) is

$$\|u_h - u\|_{-1} \leq ch^{r+1}\|u\|_r, \tag{6.4}$$

if $u \in H^r$. That is, the highest order of convergence obtainable in any norm is $\mathcal{O}(h^{r+1})$, compared with $\mathcal{O}(h^{2r+1})$ for the standard Galerkin method (see (5.11)). The nonstandard Galerkin method to which the collocation method is equivalent has as inner product the Sobolev inner product $(\cdot, \cdot)_{r/2}$, defined by (3.2). The Arnold and Wendland (1983) analysis also handles more general pseudo-differential operators L, provided that the bilinear form $(L\phi, \psi)_{r/2}$ is coercive with respect to an appropriate norm. For pseudo-differential operators of order β the highest order convergence result they obtain is

$$\|u_h - u\|_\beta \leq ch^{r-\beta}\|u\|_r, \tag{6.5}$$

compared to

$$\|u_h - u\|_{-r+\beta} \leq ch^{2r-\beta}\|u\|_r$$

for the standard Galerkin method.

A generalization this above approach to piecewise-linear collocation on the torus has been given by Hsiao and Prössdorf (1992).

The present theoretical situation for the collocation method is much less satisfactory for approximation by piecewise-constants, or other splines of even degree. For the very special case of smooth plane curves, smoothest splines and a uniform mesh, a satisfactory analysis has been developed (de Hoog, 1974; Saranen and Wendland, 1985; Arnold and Wendland, 1985; Saranen, 1988) by the use of Fourier series methods, combined with localization arguments. In this analysis the collocation points must always be chosen in an appropriate way; for a full discussion of the correct choice see Wendland (1990). For example, for the case of the logarithmic-kernel equation with a uniform mesh and smoothest splines of even degree, the collocation points should be taken to be the midpoints of each sub-interval. In this case it is known (Saranen and Wendland, 1985; Arnold and Wendland, 1985) that the result (6.4) holds. More surprisingly, in this even degree case an order of convergence one power of h higher can be obtained if u has the

appropriate smoothness: Saranen (1988) shows that

$$\|u_h - u\|_{-2} \leq ch^{r+2}\|u\|_{r+1}, \tag{6.6}$$

if $u \in H^{r+1}$.

In the following section we shall see that the collocation method is a special case of the so-called 'qualocation' method. At that point we will demonstrate the Fourier series method of analysis. A generalization of the Fourier series techniques to the case of equations on a torus has been given by Costabel and McLean (1992).

Another situation for which a reasonably satisfactory collocation analysis exists is that of singular integral operators, or systems of singular operators, on plane curves; see Prössdorf and Schmidt (1981), Prössdorf and Rathsfeld (1984), and, for an overview, Prössdorf (1989). Here the principal tool is a localization technique, combined with the observation that translationally invariant operators yield circulant matrices (in the case of closed smooth curves) or Toeplitz matrices (in the case of open arcs). The circulant matrix methods are closely related to the Fourier series techniques mentioned earlier. A generalization to multi-dimensional equations is given by Prössdorf and Schneider (1991).

In spite of the successes in the analysis of the collocation method, there remain some large gaps. Most strikingly, there is apparently no analysis as yet of piecewise-constant collocation for the single-layer equation (2.5) on a three-dimensional sphere. Fourier series methods have no obvious extension, because on a sphere there is no such thing as a uniform partition. In the two-dimensional analogue, however, significant progress with nonuniform partitions has recently been achieved: Chandler (1989, 1990, 1991) has shown for the logarithmic-kernel integral equation that piecewise-constant collocation at the midpoints is stable and convergent even for an essentially arbitrary mesh. Chandler's analysis exploits the specific structure of the collocation matrix for this problem.

7. The qualocation and related methods

In this section we consider the qualocation method (Sloan, 1988; Sloan and Wendland, 1989; Chandler and Sloan, 1990), and its fully discrete variants (Sloan and Burn, 1991; Saranen and Sloan, 1992). For a review with a more limited focus but some more details, see Sloan (1992). An earlier review, restricted to the qualocation method, is that of Wendland (1989).

7.1. The qualocation method

The qualocation method (or 'quadrature-modified collocation method') is an approximation which aims to achieve an order of convergence better than that of the collocation method, while not being too much more expensive to

implement. Significant results are so far available only for boundary integral equations on smooth plane curves, thus we shall assume that the equation to be solved is a single equation of the form

$$Lu = f, \tag{7.1}$$

where u and f are 1-periodic functions on $\mathbb{R}$.

The qualocation method is characterized by three things: a trial space S_h, which is the finite-dimensional space within which the approximate solution is to be sought; a test space T_h of the same dimension as S_h; and a quadrature rule Q_h. Given these three ingredients, the method is: find $u_h \in S_h$ such that

$$(Lu_h, \chi)_h = (f, \chi)_h \qquad \forall\, \chi \in T_h, \tag{7.2}$$

where

$$(v, w)_h = Q_h(v\overline{w}). \tag{7.3}$$

Letting $\{\phi_1, \ldots, \phi_N\}$ be a basis for S_h and $\{\chi_1, \ldots, \chi_N\}$ a basis for T_h, the equations to be solved in practice are

$$\sum_{j=1}^{N} (L\phi_j, \chi_k)_h a_j = (f, \chi_k)_h, \qquad k = 1, \ldots, N. \tag{7.4}$$

The method is in effect a semi-discrete version of the Petrov–Galerkin method, i.e. the Galerkin method with different test and trial spaces. It reduces to the Petrov–Galerkin method if $(\,\cdot\,,\,\cdot\,)_h$ is replaced by the exact inner product $(\cdot, \cdot)$. The novel feature of the qualocation method lies in the discretization: for we shall see that the recommended quadrature rules can be curious indeed.

First, though, we note the important fact that the qualocation formalism includes the collocation method as a special case. For if the quadrature rule is

$$Q_h g = \sum_{\ell=1}^{N} w_\ell g(t_\ell), \tag{7.5}$$

an N-point quadrature rule with nonzero weights $w_1, \ldots, w_N$, then (7.2) is equivalent to

$$\sum_{\ell=1}^{N} w_\ell [Lu_h(t_\ell) - f(t_\ell)]\overline{\chi}_k(t_\ell) = 0, \quad k = 1, \ldots, N,$$

which is in turn equivalent to the collocation equations (6.2) if the $N \times N$ matrix $\{\overline{\chi}_k(t_\ell)\}$ is nonsingular. It is easy to see that quadrature rules with fewer than N points necessarily make the matrix in (7.4) singular, thus only quadrature rules with N or more points are of interest.

At the present time theoretical results are available only if S_h is a space of smoothest splines of order $r, r \geq 1$, and the partition Π_h is uniform, with $h = 1/N$. For the trial space T_h Sloan (1988) and Sloan and Wendland (1989) used a trigonometric polynomial space,

$$T_h = \text{span}\left\{e^{2\pi ijx} : -\frac{N}{2} < j \leq \frac{N}{2}\right\}.$$

This choice was inspired by the Arnold and Wendland (1985) analysis of the collocation method, in which, effectively, the collocation method was treated as a qualocation method with an N-point rectangle rule for Q_h. Here, however, we shall follow Chandler and Sloan (1990) in taking $T_h = S_h'$, the space of smoothest splines of order $r', r' \geq 1$, on the partition Π_h. In practice a low-order spline test space is likely to be preferred over the trigonometric polynomial test space, because it admits a (B-spline) basis in which each element has small support. (If $r' = 1$ the value of the piecewise-constant test function at a point of discontinuity must be understood to be the mean of the left-hand and the right-hand limits. This becomes important if a quadrature point is a point of the partition.)

Following Chandler and Sloan (1990), the operator L in (7.1) is taken to be of the form

$$L = A + B, \tag{7.6}$$

where A has as its Fourier series representation either

$$Av(x) \sim \hat{v}(0) + \sum_{k \neq 0} |k|^\beta \hat{v}(k) e^{2\pi ikx} \tag{7.7}$$

or

$$Av(x) \sim \hat{v}(0) + \sum_{k \neq 0} \text{sign}\, k |k|^\beta \hat{v}(k) e^{2\pi ikx}, \tag{7.8}$$

where $\beta \in \mathbb{R}$, and where

$$B : H^s \to H^t \quad \text{for all } s, t \in \mathbb{R}. \tag{7.9}$$

In the language of the paragraph containing (4.44), L is a pseudo-differential operator of order β and principal symbol either $|\xi|^\beta$ or $\text{sign}\xi|\xi|^\beta$. Since the principal symbol is constant, i.e. independent of x, the operator A can be represented as a convolution. If (7.7) holds then the principal symbol is even, and A is said to be even. Similarly, if (7.8) holds then A is said to be odd. An important special case is that of the logarithmic-kernel integral operator: setting $\beta = -1$ and taking the even case, the operators L, A and B defined by (3.5), (3.7–11) are exactly of the prescribed form, if the free parameter α is set equal to $e^{-1/2}$.

The quadrature rule is taken to be a composite rule of the form

$$Q_h g = h \sum_{\ell=0}^{N-1} \sum_{j=1}^{J} w_j g(\frac{\ell + \xi_j}{N}), \tag{7.10}$$

where

$$0 \le \xi_1 < \xi_2 < \cdots < \xi_J < 1, \tag{7.11}$$

and

$$\sum_{j=1}^{J} w_j = 1, \qquad w_j > 0 \quad \text{for} \quad j = 1, \ldots, J. \tag{7.12}$$

Thus Q_h is the composition, onto each subinterval of the partition, of the J-point rule

$$Qg = \sum_{j=1}^{J} w_j g(\xi_j), \tag{7.13}$$

a quadrature rule defined in $[0, 1]$.

How should the rule Q be chosen? Since the choice $J = 1$ is equivalent to a collocation method, it is natural to consider $J = 2$. Chandler and Sloan (1990) restrict attention to $J = 2$ rules that are symmetric, i.e. having the property that if ξ is a quadrature point then either $\xi = 0$, or else $1 - \xi$ is also a quadrature point with the same associated weight as ξ. There are just two kinds of symmetric rule with $J = 2$, namely

$$Qg = wg(0) + (1 - w)g(\tfrac{1}{2}), \tag{7.14}$$

where $0 < w < 1$, and

$$Qg = \tfrac{1}{2}g(\xi) + \tfrac{1}{2}g(1 - \xi), \tag{7.15}$$

with $0 < \xi < \frac{1}{2}$. The first of these is analogous to Simpson's rule, and becomes Simpson's rule if $w = \frac{1}{3}$; and the second is analogous to 2-point Gauss quadrature, and becomes so if $\xi = 0.21132\ 48654\ldots$. We shall see, however, that these are usually *not* the recommended values of w or ξ. Rather, the value of w or ξ should be the unique value that will increase the maximum order of (negative-norm) convergence. In some circumstances, for example, the recommended value of w will turn out to be $w = \frac{3}{7}$ (giving the '$\frac{3}{7}$, $\frac{4}{7}$ rule').

The next two theorems give the highest-order results obtained by Chandler and Sloan (1990) for the two kinds of quadrature rule. First we collect the main assumptions.

Assumption Y: the equation to be solved is (7.1), with L given by (7.6), (7.9), and one of (7.7), (7.8); L is one-to-one; the partition Π_h is uniform; the test space is S_h', the space of smoothest splines on Π_h of order $r' \ge 1$;

and r' has the same parity as r if A is even and the opposite parity if A is odd.

Theorem 4 Assume that Y holds, that Q is given by (7.14), with $0<w<1$, and that $r > \beta + 1$.

(i) The qualocation equation (7.2) has a unique solution $u_h \in S_h$ for all h sufficiently small.

(ii) If r and A are both even or both odd then u_h satisfies

$$\|u_h - u\|_\beta \le ch^{r-\beta}\|u\|_r. \tag{7.16}$$

It satisfies also

$$\|u_h - u\|_{\beta-2} \le ch^{r-\beta+2}\|u\|_{r+2} \tag{7.17}$$

if and only if, in addition,

$$w = \frac{2^{r-\beta-1} - 1}{2^{r-\beta} - 1}. \tag{7.18}$$

(iii) If r and A are of opposite parity then u_h satisfies

$$\|u_h - u\|_{\beta-1} \le ch^{r-\beta+1}\|u\|_{r+1}. \tag{7.19}$$

It satisfies also

$$\|u_h - u\|_{\beta-3} \le ch^{r-\beta+3}\|u\|_{r+3} \tag{7.20}$$

if and only if, in addition,

$$w = \frac{2^{r-\beta} - 1}{2^{r-\beta+1} - 1}. \tag{7.21}$$

A sketch of the proof of this theorem follows Theorem 5. A first observation about the content of the theorem is that (7.16) is the same as (6.5), the fastest convergence result obtained by Arnold and Wendland (1983, 1985) for the collocation method and L a pseudo-differential operator of order β; and the one higher order result (7.19) is the improved collocation result obtained by Saranen (1988) for the case of an even operator and odd r, already referred to in Section 6. More interestingly, we see in (7.17) or (7.20) that the maximum order of convergence jumps by yet another two if (and only if) w has the precise values specified in (7.18) or (7.21).

For the particular case of the logarithmic-kernel operator A is even and $\beta = -1$, so if r also is even then the special value of w is

$$w = \frac{2^r - 1}{2^{r+1} - 1}, \tag{7.22}$$

which yields

$$\|u_h - u\|_{-3} \le ch^{r+3}\|u\|_{r+2}, \tag{7.23}$$

whereas the best result available if w has any other value is the $\mathcal{O}(h^{r+1})$

result (6.4). For example, in the piecewise-linear case (i.e. $r = 2$) the value $w = 3/7$ yields an $\mathcal{O}(h^5)$ result, compared with $\mathcal{O}(h^3)$ for the collocation method. If r is odd then the choice

$$w = \frac{2^{r+1} - 1}{2^{r+2} - 1} \tag{7.24}$$

yields

$$\|u_h - u\|_{-4} \leq ch^{r+4}\|u\|_{r+3}, \tag{7.25}$$

whereas the best result if w has any other value is the $\mathcal{O}(h^{r+2})$ result (6.6). Thus in the piecewise-constant case (i.e. $r = 1$) the value $w = 3/7$ again yields an $\mathcal{O}(h^5)$ result, compared with $\mathcal{O}(h^3)$ for the midpoint collocation method.

We should note, though, that the higher order of convergence apparent in (7.17) or (7.20) require both higher regularity of u and a more negative norm in which to observe the error. It follows that in some applications the maximum order of convergence will not be achieved.

In the next theorem the recommended quadrature points for the rule (7.15) of 2-point Gauss type are the zeros of the function

$$G_\alpha(x) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \cos\ 2\pi nx, \tag{7.26}$$

for appropriate values of $\alpha \geq 1$. It is known that G_α has exactly two zeros on $(0, 1)$, located symmetrically with respect to the midpoint. Some values of the first zero (taken from Sloan and Wendland (1989)) are given in Table 1.

Table 1. *The unique zero of* G_α *in* $(0, \frac{1}{2})$

α	Zero of G_α
1	1/6
2	0.21132 48654
3	0.23082 96503
4	0.24033 51888
5	0.24511 88417
∞	1/4

Theorem 5 Assume that Y holds, that Q is given by (7.15) with $0 < \xi < \frac{1}{2}$, and that $r > \beta + \frac{1}{2}$.

(i) The qualocation equation (7.2) has a unique solution $u_h \in S_h$ for all h sufficiently small.

(ii) If r and A are both even or both odd then u_h satisfies (7.16). It satisfies also (7.17) if, in addition, ξ is the unique zero in $(0, \frac{1}{2})$ of $G_{r-\beta}$.

(iii) If r and A are of opposite parity then u_h satisfies (7.19). It satisfies also (7.20) if, in addition, $r' \geq 3$ and ξ is the unique zero in $(0, \frac{1}{2})$ of $G_{r-\beta+1}$.

The complete proofs of Theorems 4 and 5 are lengthy. Here we indicate only the outline, with main emphasis on the argument that determines the special values of w or ξ. (For a more complete sketch of that part of the argument, see Sloan (1992).)

The main task is to prove the theorems for the special case $L = A$, the result then being extended to the full operator $L = A + B$ by a standard perturbation argument, given, for example, by Arnold and Wendland (1985). Because A is a convolution operator, and so invariant under translation, and because also the partition is uniform, the qualocation matrix in (7.4) can be made diagonal if the basis functions of S_h and S'_h are chosen so as to behave in an appropriate way under translation by h. An appropriate basis for S_h is $\{\psi_\mu : \mu \in \Lambda_N\}$, where

$$\Lambda_N = \left\{\mu \in \mathbb{Z} : -\frac{N}{2} < \mu \leq \frac{N}{2}\right\}, \tag{7.27}$$

and

$$\psi_\mu(x) = \begin{cases} 1, & \mu = 0, \\ \sum\limits_{k \equiv \mu} (\mu/k)^r \mathrm{e}^{2\pi i k x}, & \mu \in \Lambda_N^*. \end{cases} \tag{7.28}$$

Here $\Lambda_N^* = \Lambda_N \backslash \{0\}$, and $k \equiv \mu$ means that $k - \mu$ is a multiple of N. (If $r = 1$ the Fourier series, which is then not absolutely convergent, is to be understood as the limit of the symmetric partial sums.) That ψ_μ really is a spline of order r on the uniform partition Π_h follows from the fact that the Fourier coefficients satisfy the appropriate recurrence relation for a function $v \in S_h$, namely (Arnold (1983), extending Quade and Collatz (1938))

$$k^r \hat{v}(k) = \mu^r \hat{v}(\mu) \qquad \text{if } k \equiv \mu.$$

Since $\hat{\psi}_\mu(\nu) = \delta_{\mu\nu}$ for $\mu, \nu \in \Lambda_N$, the expansion coefficients of $v \in S_h$ in terms of $\{\psi_\mu\}$ are just the Fourier coefficients; that is

$$v = \sum_{\mu \in \Lambda_N} \hat{v}(\mu)\psi_\mu \quad \text{for } v \in S_h. \tag{7.29}$$

The function ψ_μ is, in essence, the spline equivalent of the trigonometric polynomial $\mathrm{e}^{2\pi i \mu x}$. In particular, the two functions behave in exactly the same way under translation by h.

With a basis $\{\psi'_\mu : \mu \in \Lambda_N\}$ for S'_h defined in a similar way, and with the aid of the expression (7.7) or (7.8) for A, it is a straightforward if tedious

matter to evaluate the matrix element $(A\psi_\nu, \psi'_\mu)_h$ explicitly, and to verify that it vanishes for $\mu \neq \nu$. In detail, we find (Chandler and Sloan, 1990, Lemma 1)

$$(A\psi_\nu, \psi'_\mu)_h = \begin{cases} 1 & \text{if } \mu = \nu = 0, \\ (\operatorname{sign}\mu)|\mu|^\beta D(\mu h) & \text{if } \mu = \nu \in \Lambda_N^*, \\ 0 & \text{if } \mu \neq \nu, \end{cases} \tag{7.30}$$

where the factor $(\operatorname{sign}\mu)$ in this equation is present only if A is odd, and where

$$D(y) = \sum_{j=1}^{J} w_j[1 + \Omega(\xi_j, y)]\left[1 + \overline{\Delta'(\xi_j, y)}\right], \tag{7.31}$$

with

$$\Delta'(\xi, y) = y^{r'} \sum_{\ell \neq 0} \frac{1}{(\ell + y)^{r'}} e^{2\pi i \ell \xi}, \tag{7.32}$$

and with

$$\Omega(\xi, y) = |y|^{r-\beta} \sum_{\ell \neq 0} \frac{1}{|\ell + y|^{r-\beta}} e^{2\pi i \ell \xi} \tag{7.33}$$

if r and A are both even or both odd, or

$$\Omega(\xi, y) = \operatorname{sign} y |y|^{r-\beta} \sum_{\ell \neq 0} \frac{\operatorname{sign}\ell}{|\ell + y|^{r-\beta}} e^{2\pi i \ell \xi} \tag{7.34}$$

if r and A are of opposite parity.

Since we will have to divide by $D(\mu h)$, for stability of the method it is essential that $D(y)$ be bounded away from zero for $y \in [-\frac{1}{2}, \frac{1}{2}]$ – a property that is not quite trivial, since it is well known that some collocation methods (e.g. midpoint collocation if r and A are both even) are unstable. Nevertheless it is shown in Chandler and Sloan (1990), by appeal to known properties of trigonometric sums, that under the conditions of the theorem there exists $d > 0$ such that

$$|D(y)| \geq d \qquad \text{for all } y \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

It now follows from the qualocation equation (7.4) and from (7.29) and (7.30) that

$$\hat{u}_h(\mu) = \begin{cases} (Au, \psi'_0)_h & \text{if } \mu = 0, \\ \dfrac{(\operatorname{sign}\mu)}{|\mu|^\beta D(\mu h)} (Au, \psi'_\mu)_h & \text{if } \mu \in \Lambda_N^*. \end{cases} \tag{7.35}$$

After evaluating the right-hand side (using e.g. Chandler and Sloan (1990,

Lemma 1)), we find

$$\hat{u}_h(\mu) - \hat{u}(\mu) = \begin{cases} P_h, & \mu = 0, \\ -[E(\mu h)/D(\mu h)]\hat{u}(\mu) + R_h(\mu), & \mu \in \Lambda_N^*, \end{cases} \tag{7.36}$$

where

$$E(y) = \sum_{j=1}^{J} w_j \Omega(\xi_j, y) \left[1 + \overline{\Delta'(\xi_j, y)}\right], \tag{7.37}$$

$$|P_h| \leq \sum_{n\equiv 0}{}' |n|^\beta |\hat{u}(n)|, \tag{7.38}$$

$$|R_h(\mu)| \leq \frac{c}{|\mu|^\beta} \sum_{n\equiv\mu}{}' |n|^\beta |\hat{u}(n)|, \tag{7.39}$$

and where

$$\sum_{n\equiv\mu}{}' = \sum_{\substack{n\equiv\mu \\ n\neq\mu}}.$$

The error expression (7.36) is the key to the theorems. The quantities P_h and $R_h(\mu)$ in the expression depend only on the Fourier coefficients $\hat{u}(n)$ for which $|n| \geq N/2$, and so can be made to decay as rapidly as desired by requiring u to be in a sufficiently high Sobolev space. The first term for $\mu \in \Lambda_N^*$ is in a quite different category, because it is this that imposes an absolute restriction on the maximum order of convergence that can be achieved: if $E(y) = \mathcal{O}(|y|^\rho)$ as $y \to 0$ then the best order of convergence we can hope for, given (7.36), is $\mathcal{O}(h^\rho)$. For this reason ρ is called by Chandler and Sloan (1990) the 'order' of the particular qualocation method.

If r and A are both even or both odd then it follows from (7.32), (7.33) and (7.37) that, for any symmetric rule Q,

$$E(y) = |y|^{r-\beta} 2 \sum_{j=1}^{J} w_j \sum_{\ell=1}^{\infty} \frac{\cos 2\pi\ell\xi_j}{\ell^{r-\beta}} + \mathcal{O}(|y|^{r-\beta+2}). \tag{7.40}$$

(Note that $r' \geq 2$, since under the present assumptions r' is even.) Thus the qualocation method is of order $r - \beta$ – unless, that is

$$\sum_{j=1}^{J} w_j \sum_{\ell=1}^{\infty} \frac{\cos 2\pi\ell\xi_j}{\ell^{r-\beta}} = 0, \tag{7.41}$$

in which case the order jumps to $r - \beta + 2$. For a rule Q of the form (7.14) the latter equation becomes

$$w \sum_{\ell=1}^{\infty} \frac{1}{\ell^{r-\beta}} + (1-w) \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell^{r-\beta}} = 0,$$

or by a standard zeta function trick

$$\left[w - (1-w)(1 - \frac{1}{2^{r-\beta-1}})\right] \sum_{\ell=1}^{\infty} \frac{1}{\ell^{r-\beta}} = 0,$$

which is satisfied if and only if w has the value given by (7.18). And for a rule Q of the form (7.15) it is immediately obvious that (7.41) is satisfied if and only if ξ is the zero in $(0, \frac{1}{2})$ of the function $G_{r-\beta}$ defined by (7.26).

Similarly, if r and A are of opposite parity (and hence r' is odd) then it follows from (7.32), (7.34) and (7.37) that, for Q a symmetric rule,

$$E(y) = -|y|^{r-\beta+1} 2(r-\beta) \sum_{j=1}^{J} w_j \sum_{\ell=1}^{\infty} \frac{\cos 2\pi\ell\xi_j}{\ell^{r-\beta+1}} + \mathcal{O}\Big(|y|^{r-\beta+\min(r',3)}\Big).$$

Thus the method is of order $r - \beta + 1$, unless

$$\sum_{j=1}^{J} w_j \sum_{\ell=1}^{\infty} \frac{\cos 2\pi\ell\xi_j}{\ell^{r-\beta+1}} = 0, \tag{7.42}$$

in which case it is of order $r - \beta + 3$, provided $r' \geq 3$. The special values of w or ξ in rules of the form (7.14), (7.15) are as before, but with r replaced by $r + 1$.

The orders of convergence in every case are now seen to correspond exactly to the maximum orders of convergence in Theorems 4 and 5. For the remainder of the proof we refer to Chandler and Sloan (1990). In particular, Theorem 2 of that paper shows that all the results follow once the order and stability of the method has been established.

The qualocation analysis indicated above conforms to one of the great paradigms of numerical analysis: first identify the form of the leading term of the errors, then adjust the method so as to eliminate that leading term. Looked at that way, the 3/7, 4/7 qualocation rule is no stranger than, say, the formulas of Romberg integration.

7.2. Fully discrete variants

We have seen that the Galerkin method for (7.1) requires two levels of integration for each matrix element, whereas the collocation and qualocation methods need only one level. But the following variant of the qualocation method proposed by Sloan and Burn (1991) for the logarithmic-kernel integral equation on a smooth curve requires no exact integrals at all.

In this method, the exact integral (3.5) is first replaced by its rectangle rule approximation

$$L_h u(x) = -2h \sum_{k=0}^{N-1} \log|\nu(x) - \nu(kh)| u(kh). \tag{7.43}$$

Then one proceeds as in the qualocation method: find $u_h \in S_h$ such that

$$(L_h u_h, \chi)_h = (f, \chi)_h \qquad \forall\, \chi \in T_h. \tag{7.44}$$

Here $(\cdot, \cdot)_h$ is defined again by (7.3) and (7.10–13), but now the parameters in the rule Q must be chosen differently, as the quadrature rule has the added burden of compensating for the damage caused by replacing L by L_h.

In working out this method u_h is evaluated only at the points of the rectangle rule (7.43), thus the trial space S_h becomes significant only if one wants to interpolate between the points. For the analysis, however, the choice of trial space is important. In Sloan and Burn (1991) a trigonometric trial space

$$S_h = \{e^{2\pi i \mu x} : \mu \in \Lambda_N\} \tag{7.45}$$

was assumed.

The following result was established in Sloan and Burn (1991) for the case of a circle by Fourier methods similar to those used above, but for general smooth curves was proved only under additional restrictions. The result for general curves was proved without the extra restrictions by Saranen and Sloan (1992).

Theorem 6 Assume that the equation to be solved is $Lu = f$, where L is the logarithmic-kernel integral operator in (3.5); that the transfinite diameter is different from 1, so that L is one-to-one; that the partition Π_h is uniform; that the trial space is given by (7.45); that the test space is S'_h, the space of smoothest splines of order r'; that r' is even; and that

$$Qg = \tfrac{1}{2} g(\xi) + \tfrac{1}{2} g(1 - \xi),$$

with $0 < \xi < \frac{1}{2}$. Then

(i) Equation (7.44) has a solution $u_h \in S_h$ for all h sufficiently small.

(ii) For $s \geq -1$, u_h satisfies

$$\|u_h - u\|_s \leq c h^1 \|u\|_{s+1}. \tag{7.46}$$

It satisfies also

$$\|u_h - u\|_s \leq c h^3 \|u\|_{s+3} \tag{7.47}$$

if and only if $\xi = \frac{1}{6}$.

Versions with maximum order higher than $\mathcal{O}(h^3)$ have been foreshadowed in Sloan (1992).

An alternative version proposed by Saranen and Sloan (1992) replaces the right-hand side of (7.44) by the exact inner product. Thus the method becomes: find $u_h \in S_h$ such that

$$(L_h u_h, \chi)_h = (f, \chi) \qquad \forall\, \chi \in T_h. \tag{7.48}$$

This has an advantage if u is of low regularity, in that the condition $s \geq -1$ in Theorem 6 is replaced by $s \geq -r' - 1$. On the other hand the estimate (7.47) is replaced by the more restrictive estimate

$$\|u_h - u\|_s \leq ch^{\min(r',3)} \|u\|_{s+\min(r',3)}. \tag{7.49}$$

8. Corners, cracks and challenges

At many points we have mentioned difficulties caused by corners. Part of the problem is that the techniques of analysis (e.g. pseudo-differential operator arguments, compactness of operators, Fourier series methods) break down when corners are present. Part of it is that corners force us to consider modifications (such as mesh grading near the corner) which further complicate the analysis. Sometimes (as in the case of the double-layer equations) the presence of corners forces changes in a method (such as modifications in the trial space for the collocation method near a corner), even though there is little or no evidence that such changes are needed other than to make the proofs go through. It is fair to say that even for plane problems corners still present many theoretical challenges.

Consider, for a moment, the qualocation method and its discrete variants, described in Section 7. Once corners are present the Fourier series arguments outlined there break down, because it is no longer possible to write the boundary integral operator in the form $L = A + B$ with A given by (7.7) or (7.8) and B a smoothing operator as in (7.9). Yet numerical experiments (Chandler and Sloan, 1990; Sloan and Burn, 1991) suggest very strongly that the methods can remain useful, and even yield orders of convergence similar to those predicted for smooth curves, if the mesh is suitably graded in a neighbourhood of the corner.

Curiously, there is one extreme case of a corner, namely the exterior Dirichlet problem for a slit or crack, for which the theoretical understanding is reasonably complete. Taking for simplicity a straight slit of length 2γ, and forgetting the boundary condition (2.7) at infinity, the single-layer equation (2.5) becomes

$$-\frac{1}{\pi}\int_{-\gamma}^{\gamma} \log|t-s| z(s) \mathrm{d}s = g(t), \quad t \in (-\gamma, \gamma). \tag{8.1}$$

Applying the transformation (Yan and Sloan, 1988)

$$t = \gamma \cos 2\pi x, \qquad s = \gamma \cos 2\pi y, \tag{8.2}$$

$$f(x) = g(\gamma \cos 2\pi x), \tag{8.3}$$

$$u(x) = \gamma z(\gamma \cos 2\pi x) |\sin 2\pi x|, \tag{8.4}$$

we obtain (on noting that f and u are even and 1-periodic)

$$-\int_0^1 \log|\gamma(\cos 2\pi x - \cos 2\pi y)|u(y)\mathrm{d}y = f(x), \quad x \in \mathbb{R}. \tag{8.5}$$

The cosine transformation used here has a long history, particularly in connection with the airfoil equation (Multhopp, 1938; Weissinger, 1950; Schleiff, 1968a,b). Its particular advantage in the present context is that it reduces the problem to one we have met already, namely the logarithmic-kernel equation for a circle: using only trigonometric and logarithmic identities and the fact that u is even, it can easily be shown (see Prössdorf *et al.* (1992), Lemma 2.1) that (8.5) is equivalent to

$$-2\int_0^1 \log|(2\gamma)^{1/2}\sin\pi(x-y)|u(y)\mathrm{d}y = f(x), \quad x \in \mathbb{R}. \tag{8.6}$$

But from (3.7) this is just the single-layer integral equation for a circle of radius $(\gamma/2)^{1/2}$.

The transformations leading to (8.6) tell us that the solution z of (8.1) usually has singularities of the form $(\gamma \mp t)^{-1/2}$ at the two ends (this follows from (8.4)). It also tells us, since a circle has a transfinite diameter equal to its radius (see Subsection 4.3), that equation (8.1) for z is singular when $(\gamma/2)^{1/2} = 1$, or $\gamma = 2$. (This corresponds to the fact, mentioned in Subsection 4.3, that the transfinite diameter of an interval is one quarter of its length.)

More importantly for our present purposes, this transformation lies at the heart of several theoretical analyses of numerical methods for the logarithmic-kernel integral equation on open arcs. These include Atkinson and Sloan (1991), which gives an analysis of a discrete Galerkin method; Sloan and Stephan (1992), analysing a collocation method with Chebyshev polynomials; Prössdorf *et al.* (1992), adapting to an open arc the discrete method of Sloan and Burn (1991) discussed in Subsection 7.2; and Joe and Yan (1991, 1992), analysing a piecewise-constant collocation method on a graded mesh, with the collocation points taken to be the midpoints with respect to the transformed variable x in (8.2) rather than with respect to the original variable t.

That work of Joe and Yan (1991, 1992) establishes the (surprising) conclusion that the order of convergence can be increased by a seemingly insignificant shift in the collocation points. Indeed, the order of convergence established by Joe and Yan is even higher than the apparent order of convergence of the Galerkin method for the same piecewise-constant basis and the same graded mesh (Yan and Sloan, 1989). A lesson for the future seems to be that in both the theory and the practice of mesh grading we need to take more seriously than in the past the transformed independent variable implicit in the mesh grading: if the partition is uniform with respect to the

transformed variable x, then perhaps we should specify all collocation points and quadrature rules with respect to that variable.

A serious challenge for many boundary element methods is the extension of the analysis to irregular meshes on plane curves, and thence to three-dimensional surfaces, which is after all where the main game is. This certainly poses a problem for methods such as the qualocation method which rely on Fourier series methods for their analysis. It is even a problem, as we have remarked before, for a method as simple as piecewise-constant collocation for the single-layer equation on a sphere.

There is still much to be done. Those of us who enjoy the field are happy that this is so.

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